Definition: A UPC has exactly 1 cycle of length 3 to n. For example, the most trivial case of a UPC is a triangle, since it has exactly one cycle of length 3.

We aren't sure how many UPCs exist, since they don't exist on all numbers of vertices (for example, 14 vertices). Someone tried to find examples of UPCs and his computer crashed at 60-something vertices. Since UPCs can't always be represented as graphs, we extend the definition of UPC graphs to UPC matroids: $M(E, c)$, where c is a collection of circuits.

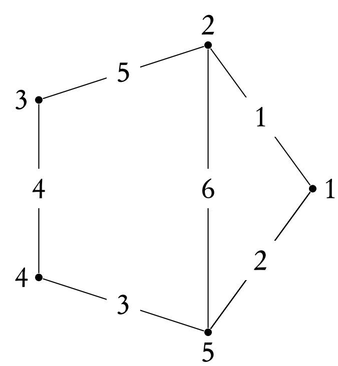
Definition A *circuit* is a minimally dependent set, just as cycles are minimally dependent vines of graphs. Minimally dependent means that, if we take any element out of the circuit, we are left with an independent set.

Using properties of matroids, we can write the following:

- $\varnothing \in e$
- if $c_1, c_2 \in e, c_1 \leq c_2 \implies c_1 = c_2$
- $e \in c_1 \cap c_2, \exists c_3 \leq c_1 \cup c_2 - e$

Last Monday, we were also able to use incidence matrices convert from matroids to matrices. Before we started working on such conversions, however, Corrine introduced some new notation.

Definition A *basis* is a maximally independent set, and the *rank* is the size of a basis. We showed that the rank of a UPC with n elements will always be $n - 1$. We will then use the notation UPC(n) to describe a unipancyclic graph with $n + 1$ elements. We can also call the above graph a rank-n UPC.



Nathan came up with the following matrix:

rank 4	1	2	3	4	5	6
1	1	0	0	0	1	1
2	0	1	0	0	1	1
3	0	0	1	0	1	0
4	0	0	0	1	1	0

which is the rank-4 UPC expressed in binary. The columns 1 through 4 are just an identity matrix. They represent the standard basis vectors. Columns 5 and 6 represent the edges we add to make the matroid a

UPC. Column 5 represents the 5th neck, and column 6 represents the 6th neck. We can do the same thing for a rank-7 UPC matrix. Such a matrix takes too much space, so it's not included in these notes, but if you would like to make one, go ahead.

As it turns out, being able to express a matroid as a graph means that it can be expressed as a matrix in any base. For this reason, UPC-24 cannot be expressed as a graph. Even though it can be expressed as a binary matrix, it cannot be expressed as a ternary matrix.

2.3 Wednesday – Can you hear the shape of a windchime? With Max, AI. (Max, student)

Bees start in a set of beehives and at every turn migrate to one of the closest hives with equal probability.

The map that takes the initial distribution to the distribution after n turns is linear.

The n-turn matrix for this graph is

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix}^n$$

This is the transition matrix and is the inverse of the diagonal matrix, the matrix resulting from writing the degree of each vertex in the diagonal, multiplied by the adjacency matrix.

Nathan conjectures that is a graph G is not bipartite then in the limit of proportion of bees at a vertex v is $\frac{\deg(v)}{2e}$

Define $\Delta = \{v \in \mathbb{R}^n | v_i \geq 0, \sum_{i=1}^n v_i = 1\}$ and consider a map $F : \Delta \to \Delta$ where $\vec{v} \to \dfrac{T\vec{v}}{T\vec{v} \cdot (1, 1, 1, \ldots, 1)}$

Claims:

1. $F(\Delta) \subset \Delta$
2. F is continuous
3. if V is a fixed point then $T(\vec{v}) = \vec{v}$

$F : \Delta \to \Delta$ is a continuous map so by Brower's fixed point theorem $\exists x \in \Delta$ s.t. $F(x) = x$. This means that here is a fixed point of $T(\vec{v})$.

This is essentially how the earliest version of Google's pagerank worked by modeling the web in a way where each webpage was a vertex and links were directed edges. The algorithm would run bees on this graph and look at the final distribution and then returns the pages with the most bees.

Spectral graph theory and harmonic analysis on graphs looks at the question of how fast do we get to the fixed point.

Associated with the transition matrix are eigenvalues but graphs with the same spectrum can be different. This question can also be considered on surfaces with respect to differential operators like the heat equation.

2.4 Thursday – The Geometry of SET, with Liz McMahon (Kamil)

*A **wonderful** article by Kamil.*

2.4.1 Rules

The object of the game is to identify a SET of 3 cards from 12 cards placed face up on the table. Each card has four features, which can vary as follows:

A SET consists of 3 cards in which each of the cards' features, looked at one-by-one, are the same on each card, or, are different on each card. All of the features must separately satisfy this rule. In other words: the shape must be either the same on all 3 cards, or different on each of the 3 cards; color must be either the same on all 3 cards, or different on each of the 3, etc.

2.4.2 Associating vectors to cards

Each card can be assigned a vector with 4 components in $\mathbb{Z}_3$, each representing a characteristic:

- Number: $0, 1, 2 = 3, 1, 2$
- Color: $0, 1, 2 =$ green, purple, red
- Shading: $0, 1, 2 =$ empty, striped, solid
- Shape: $0, 1, 2 =$ diamond, oval, squiggle

In order for 3 cards to be a SET, the vectors of these three cards must add up to the zero vector in $\mathbb{Z}_3^4$.

2.4.3 Affine transformations

Affine transformations act on vectors in two ways: they multiply by square matrices, and translate by adding vectors.

Affine maps will always take SETs to other SETs. All SETs in SET are affinely equivalent.

The deck of SET cards are a geometry, with each card as a point. If three points are on a line, those three cards form a SET

The Fundamental Theorem of SET Two cards uniquely determines a third card which completes the SET.

2.4.4 Maximal Caps

Terence Tao, one of the leading mathematicians, asked, *How many cards you can have with no SETs?*

A *maximal cap* is a set of cards with maximum number of cards such that there is no SET. A *complete cap* is a cap that is not maximum but cannot be made larger without creating a SET.

$2, 4, 9, 20$ are the sizes of the maximal caps for respectively $1, 2, 3, 4$ characteristic SET.