and 23 dimensions. Then he introduced the formal statement to this problem which is: Among all sets $X = a, b, c$ of unit vectors in $\mathbb{R}^3$, minimize $\max\{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}$

To complicate things even more, we began to ponder on lines through the origin in three dimensions. The problem was how far "apart" can we place three lines in $\mathbb{R}^n$ such that these n lines are almost perpendicular.The shocking answer was that we can get n lines for n dimensions when $n < 58$ but when $n = 58$ we can get 59 almost perpendicular lines and from there the number grows exponentially.

Finally, we got to the fun part. Professor Martin amazed everyone when he turned the words "Lines through the origin" to "Linguinis through the oranges". Oranges and linguini were distributed and our goal was to poke linguini through the oranges and try to get the maximum amount of linguini that one can fit when each linguini is in 45 degree angles. In the end, few people succeeded, and the others either ate their oranges or created sea urchin-like objects out of artistic skill (Constantin).

We know that there can be at most n^2 equiangular lines in $\mathbb{C}^n$ and a set of these lines are called a SIC-POVM (symmetric, informationally complete, positive operator valued measure). SIC-POVMs are useful in quantum cryptography and quantum state tomography, but no one knows how to construct them! The only constructions so far are in all dimensions up to $n = 16$ and $n = 19, 24, 28, 35, 48$. With the other cases, we only have algebraic proofs.

Professor Martin then explained a proof about why we can't fit more lines until we reach $\mathbb{R}^{58}$ We find that it was very hard to get angles smaller than 89 degrees in high dimensions.

Professor Martin ended his presentation by talking about hyperovals and how the image on his shirt is a simple hyperoval puzzle.

3.2 Tuesday: Nathan
What do aliens mean by congruence?

Brian$_3$ has just arrived from an alternate universe. In this universe, polygons are considered congruent if one can be decomposed into the other. We quickly note that 2 congruent polygons must have equal area. Can we go the other way?

Question: Is congruence symmetric? (A $\sim$ B implies B $\sim$ A)? Transitive? (A $\sim$ B and B $\sim$ C imply A $\sim$ C)? Charlotte answered, "Yes, and Brian$_3$ why aren't you listening to me‽‽"

If we can prove all figures with area 1 are congruent to the unit square, then we are done.

Step 1: Triangulate the polygon

We showed this was possible by induction. The Base Case is n=3, which is tautological. For the inductive step, we simply must show that every polygon contains at least one of its diagonals. This is simple, we simply pick a vertex and draw a line between the two adjacent vertices. If there are no vertices in the created region then connect these two vertices. Otherwise, consider the "highest" vertex within the created region and draw the diagonal from this vertex to the original vertex selected.

Step 2: Go from triangles to rectangles

This can be done by choosing an acute angle of the triangle, drawing the altitude from this vertex, splitting the triangle into two right triangles, drawing a line through the midpoint of one of its legs parallel to the other leg and piecing it back together.

Step 3: Go from rectangles to rectangles with side length 1

We can split the rectangles in half until their dimensions are no more uneven than 2:1. Then, we can choose diagonals on the two rectangles that are both of length x. Then we can turn these rectangles into parallelograms, which in turn we can turn into rectangles with one side of length x. Because these new rectangles also have equal area, they are congruent.

Hence, we can convert any polygon of area 1 into rectangles with one side length equal to 1. After stacking these up, we are done.

One interesting generalization is to ask the same question in three dimensions. However, the opposite result actually holds in this case: there actually are noncongruent (In $Brian_3$'s sense of the word) polyhedra with equal volume.

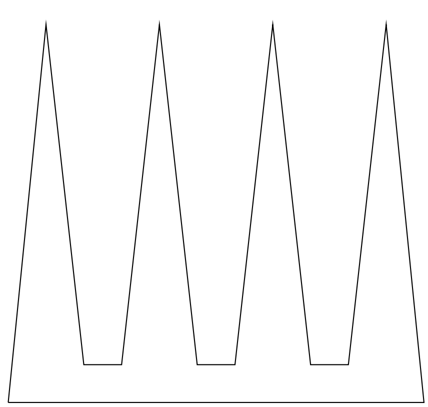
3.3 Wednesday: Alex
The Art Gallery Problem

3.3.1 Intro

During Wednesday's Daily Gather, we talked about guarding art galleries. The question goes as such: consider an art gallery with straight walls that form a polygon. We wish to place guards (who are lazy and don't move around except to turn) so that the entire art gallery is guarded at all times.

Of course, these guards can't see through walls, but they can see infinitely far. The question posed was such: consider a polygon with n sides. What is the maximum number of guards we will need to guard this polygonal-art-gallery?

The group experimented with some pre-printed shapes. We found points defined as "independent witnesses" if there is no point that a guard can stand on and see any two at the same time. After a bit of testing, Max discovered a shape with $3n$ sides that required n guards to cover.



This shape was called “Chvatal’s Comb”. It can be generalized to any number of spikes. Then, a proof was shown in class for the maximal number of required guards, called Fisk’s Proof of the Art Gallery Theorem. It can be mapped out as follows:

3.3.2 Fisk’s Proof

Fact One: Any polygon can be triangulated. We proved this in class; in general, select an acute vertex. Then, find the two nearest vertices that are along the adjacent edges. Connect those two. Either we are successful, or there is an in-jut. If there is an “injut”, take the nearest “injut” as our point. Repeat *ad infinitum.* Hand-wavy-induction-induction.

Fact Two: Every triangulated polygon can be 3-colored. There was, unfortunately, no proper proof shown in class.

We the select the smallest set of points of the same color and place guards on those points. This is the solution. It follows that we have at most $\lfloor \frac{n}{3} \rfloor$ guards required as an upper maximum.

3.3.3 Variations

Afterwards, the discussion was extended to a few (that is to say, two) variations. One of these was to consider a polygon with holes; i.e. a space on the interior that was considered outside the polygon. Dealing with holes makes the discussion a bit more challenging. A quick proof was shown that $\lfloor \frac{n+2h}{3} \rfloor$ always works. Using two additional edges, we can arbitrarily add a mini “space” in order to make the hole part of the exterior, simplifying the problem to the case of a polygon with $n + 2h$ sides.

In 3 dimensions, the discussion becomes much more challenging. It is no longer consistently possible to “tetrahedronalize” the shape, because a base case is currently unknown for the induction process. Also, placing points on all the vertices bizarrely does not guarantee a solution for some polyhedra.

3.4 Thursday: Movies

3.5 Friday: Charlotte
Lisa’s Topology Stuff

Lisa prefaced this talk by saying that we would not prove many of the concepts she would be working with.

She then described a plane P under a sphere S. The top point of S is called the north pole, or NP. She defined a function f such that if a point x on S was connected via a line to NP, $f(x)$ was the point on P that line intersected with. However, the class noted that this function was not bijective, because $f(NP)$ is not define. So we redefined f so that its domain was S^2 NP. Therefore, this function is both bijective and continuous. Lisa told the class we could use this function as a way to think about S^2 in terms of $\mathbb{R}^2$. We said that S^2 is the one-point compactification of $\mathbb{R}^2$. She then said she wanted us to try and extend this and think about S^3 as a one-point compactification of $\mathbb{R}^3$.

Next, we came up with several ways to create a sphere, and extended those ideas to try and help us picture S^3. First, Lisa described a sphere as two disks (known as D^2) glued together along their boundary. We used this to define an S^3 as two D^3 glued together along their boundary. Then, we said that a sphere is a stack of circles glued together, with points glued on at the very top and very bottom. We used this to say that an S^3 is a stack of spheres glued together, with points glued on at the top and bottom as well. Finally, Lisa told the class that an S^2 is a disk, whose boundary has been sent to a point. Therefore, you can think of an S^3 as a D^3 whose boundary has been sent to a point.

After we had done this, Lisa gave us a definition for Cartesian product. She said she wanted to think about the Cartesian product AB as "an A worth of Bs." Therefore, we can think of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an $\mathbb{R}$ worth of $\mathbb{R}$s, so every point in $\mathbb{R}^1$, the number line, has its own set of $\mathbb{R}^2$s.

Lisa then had the class try to use this definition to visualize and describe a series of Cartesian products. $S^1 \times S^1$ can be thought of as a circle on which every point has a circle coming out of it, so it is a torus. Using the same method, we were able to say that $S^1 \times D^2$ is a solid torus (a torus that is filled in the middle).

Lisa defined a spaghetti space as a space in which every point has a neighborhood looks likes a spaghetti brownie. A spaghetti brownie is a region that looks like a bundle of spaghetti or wires.

Next, Lisa drew a solid torus in S^3 and placed a series of warped disks on its top and bottom, fitted around each other like cups, as shown in Diagram 1 [1]. Because they fit around each other, they balloon outwards as you get further from the center of the torus. So, the disk at the equator, E, stretches outward like a plane, but with a hole in the middle. It is the place where two disks are attached along their boundary, since the disks on the top and the disks on the bottom meet here, so it seems as though it should be a sphere, based on what we previously said. However, this is still a disk, since a sphere with a hole cut out is a disk.

After we completed this, Lisa defined a 3-manifold to be a set M such that every point has a neighborhood that "looks like" $\mathbb{R}^3$. She also told us that one way to get a 3-manifold is by gluing together polyhedra.

Then, she drew the image depicted by Diagram 2. This depicts the creation of a shape by taking the "vertical" circles on a solid torus (as shown by A), which run along circle B, and gluing them to the "horizontal" circles on another solid torus (as shown by C), which run along circle D. This forms an S^3. This is the same setup and result as the layering of disks previously.

She also drew Diagram 3, which shows the creation of a shape by taking the "vertical" circles in one torus (defined the same way as before), and gluing them onto corresponding "vertical" circles on a second torus. This creates an $S^1 \times S^2$, a circle worth of spheres.

The last shape Lisa asked us about was a cube which was stretched so that each set of opposite faces touched. This, we decided, forms a shape $S^1 \times S^1 \times S^1$.

[1] All referenced diagrams in this section can be found on the next page.

Diagram 1

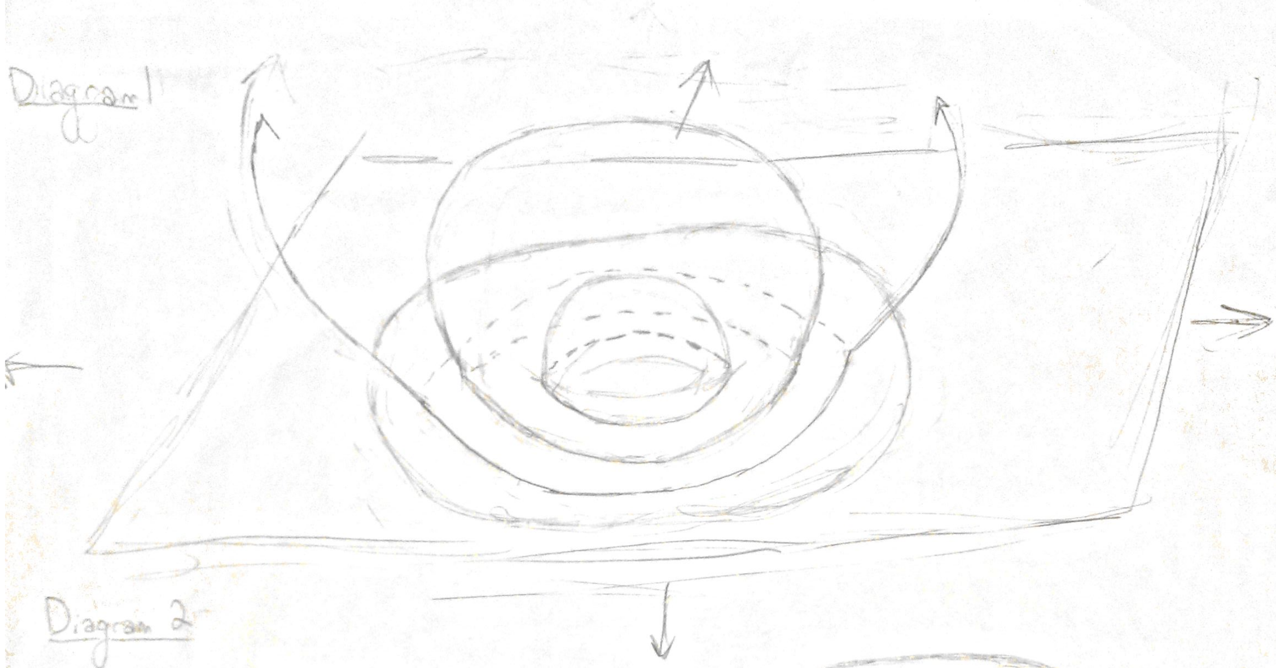


Diagram 2

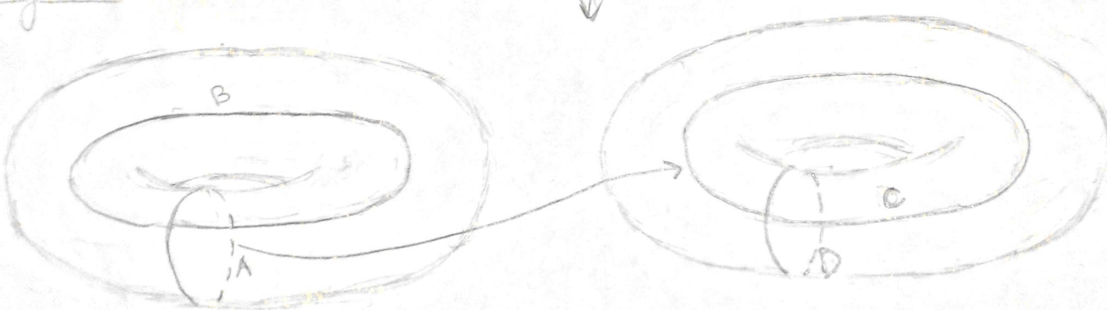


Diagram 3