2.1 Math Movies! BY Daniel & Shlok

Associativity What is associativity? Similarly, what is associative? Wikipedia defines it as "a property of some binary operations, which allows rearrangement of the parentheses." But as an average Infinite Series viewer, you are inclined to go above and beyond. You don't want to ask what is associative; you want to ask what isn't associative. The answer is: cars traveling on donuts.

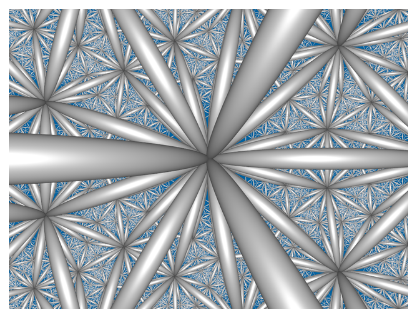
The next logical step is to make the jump to associahedrons, where the area between edges, the volume between faces, and the hypervolume between hyperplanes are all homotopies. "Why", you ask? We have no idea.

Homage to Hilbert Google just released 3-D Snake!!! With multiple playable snakes!!! With my upcoming senioritis, watch me get the high score of 4096 in 3-D Snake in the middle of AP PreCalc.

Not Knots What's a not? It's knot what you think it is. Knot making the wrong assumption is necessary with nots. The Shlomorromean Rings are the mess of nots we will consider today. Grab 3 nots and insert them such that no not can be separated from the other two. That mess of nots might knot be possible. But make it work somehow. Now, grab the cone point of their space, and stretch it to infinity. The space this will create for each Shlomorromean Rings will span all of three-dimensional space.

Now, you have a space full of Shlok. It's pretty useless.

Hyperbolic Rhombicdodecahedron It bends light. And Water. And Earth. And Fire. And Air.



The Divided Man

- *"That's a man!! I thought it was an elephant!"*
  —Clàudia
- *"In what mental state do you have to be in to do that?"*
  —Audrey

Somehow, this is the most heart-wrenching math movie by far.

Geodesics The trails were vibrating on the platycodons. Robert was vibrating on his water bottle during his sleep.

Rotating Rooks Place a rook on a sphere. Moving it around causes it to rotate. This doesn't hold true on a chessboard.

If you place it on a Tworus, then it will also rotate. But in the opposite direction. Now, try this on a Mhorus. To create it, use the leftover donuts from Gabe's Birthday; there are still eight dozen left.

Danlein Bottle Commence Explosions. Cue Loud Music. Display Self-Intersections. Competing for the Danius Strip in the Uncle Danny's Grand Prix, you hear Uncle Danny explain the shape of his platycodon/track using his native Russian. Vroom Vroom… Zoom Zoom…

2.2 In a world where friends of your friends are also your friends, Möbius came to invert it all. BY Audrey

Rafa from Loyola University Chicago visited us on Tuesday armed with math memes and posets. A poset, or partially ordered set, is a set P together with a relation $\leq$ that is:

- Reflexive: $x \leq x$ for all $x \in P$. AKA I am my own friend!
- Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in P$. AKA Friends of my friends are my friends!
- Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$. AKA My only true friend is myself! (...antisocial?)

We proceeded to discuss and draw diagrams depicting posets of subsets $\mathcal{B}_n$, chains $\mathcal{C}_n$, divisors $\mathcal{D}_n$, and partitions Π_n. We also discussed isomorphic posets; Rafa showed us that since posets are products of chains, $\mathcal{B}_n \cong C_2 \times \cdots \times C_2$ where C_2 is multiplied n times.

After familiarizing ourselves with properties of posets, Rafa had us play a simple recursive game. We computed the Möbius values in some example posets, using our two recursive definitions: $\mu(x, x) = 1$ and $\sum_{x \leq z \leq y} \mu(x, x) = 0$ for $x < y$.

After this introduction to posets, Rafa showed us some really cool connections to other topics we have been learning! The patterns introduced in this game come up in problems that may seem completely unrelated, such as graph colorings, hyperplanes, and topology.

In proper colorings of graphs, we can define the chromatic polynomial $\chi_G(x)$ of a graph G as the number of ways we can properly color G with at most x colors. We found a surprising correlation between the coefficients of the polynomial and the Möbius values of a poset of partitions.

Moving on to the next connection—hyperplanes in a vector space—we found we could create a poset to reflect the intersections of the hyperplanes. Rafa introduced a theorem that uses this poset to calculate the number of regions and bounded regions in the hyperplane arrangement!

Lastly, posets can be used in geometry and topology as well! Rafa introduced us to the order complex of a poset, and we calculated the reduced Euler characteristic, a quantity preserved after "bending and stretching space", of some complexes. Like the other connections, we found a cool relation to the uppermost Möbius value in a poset.

And that's a wrap on Rafa's amazing Daily Gather!

2.3 Goat Archeology: Gone But Not Forgoatten BY Noam

Ever wondered where the MathILy-EST students go while we have our classes? Well, Charlie has finally revealed the answer. These budding archaeologists have been flying in Brian's private jet through the Carpathian Mountains of Romania, discovering the secrets of an ancient goat society. Charlie dazzled us with his stories of the tablets MathILy-EST has found, with images of the marvelous goat cities from the days of yore. These cities have mountains connected by goat paths, which Charlie and his colleagues have been studying. These goat paths follow three rules:

1. Every two mountains have exactly one goat path between them.
2. Every goat path passes through the same number of mountains.
3. Every mountain has the same number of goat paths through it.

There are two types of cities, Eniffa and Jőrp, which MathILy-EST finds particularly interesting, subject to these rules:

Eniffa
- For every goat path G and a mountain $M \notin G$, there exists a (unique) goat path G' such that $M \in G'$, and $G \cap G' = \emptyset$.
- There exist 3 mountains not all on one goat path.

Jőrp
- Every 2 goat paths meet at exactly one mountain.
- There exist 4 mountains, no 3 of which are on the same goat path.

We found some examples of both Eniffa and Jőrp cities, and tracked the number of paths per mountain and number of mountains per path for each example city. For Jőrp cities, these two values were suspiciously the same.

Bella then explained the terrible predicament of these goat cities. There was no lighting, and so all the goats were tripping off of cliffs! Bella's plan was to light up all the paths using watchtowers while minimizing the number of watchtowers placed. We worked on this for a while, and Alo showed us that in any Jőrp we can always light up an entire city by placing our watchtowers along every mountain contained in one goat path. We were left with a question: how do we prove this configuration is minimal?

It appeared that we had found a way to light up any goat city we pleased, but then Shari delivered some more terrible news: the goat economy had collapsed! Now, instead of watchtowers, they only have flashlights to illuminate their paths. We want to find out if a well-lit city is still possible. According to Jaedon, in any Jőrp city, it is! And now our extremely frugal goats can live happily ever after!

Ethan then explained MathILy-EST's exploration into probabilistic city planning and the mysterious number q, which is related to the magic pouches from Frank's Daily Gather. q appears in a bunch of formulas related to these goat cities, and MathILy-EST has been hard at work trying to find the probability that a city is well-lit. This involves drawing curves through the treacherous goat mountains and finding tangency lines and singularities, which are where a curve passes through the same point twice. These tangencies correspond to flashlights and the singularities correspond to watchtowers, making them extremely useful to determine the probabilities our archaeologists yearn for.

Anna then explained how they've found upper and lower bounds for these probabilities in Eniffa cities with one watchtower. These bounds are given by the very simple formulas q^{-q^2} and $e^{-q^2-1}q^{-q^2}(1-q^{-1})^{q^2-1}$, respectively. Anna showed us how even for small values of q, these give extremely small probabilities. Ren followed up by presenting the city planners' calculations for the lower bound and upper bound of probabilities in cities with at least one watchtower. This involved a lower bound of $(q^2+q+1)q^{-3(q+1)} -$

$\binom{q^2+q+1}{2} q^{-3(2q+1)}$ and an even messier upper bound, but for large and complex cities (the ones MathILy-EST cares most about) the actual value is close to q^{-3q-1}.

Vera concluded the Daily Gather by describing the work MathILy-EST has been doing for NAGA, the National Association of Goat Astronomy. Here, our archaeologists have planes and lines that are tangent to surfaces. In this strange new world of the space goats, we can have far weirder surfaces, with far weirder results. We wondered at the might of MathILy-EST's pioneers in archaeology and goat astronomy when Vera showed us how to take an Eniffa city and crush it into a ball. What was once a plane was now a sphere. What was once a line was now a circle. What was once a point…was still a point.

Vote Goat!

2.4 I am average at chess BY Sophia J

Does anybody know how to play chess? Or rather, does anybody not know how to play chess? Brian says he hasn't played in 12 years. Rather than remember where all the pieces can move, he will place only queens (the best piece, and he will not take any objections because who's giving the Daily Gather, him or you?) on the board. If we placed the queens along the main diagonal (top left to bottom right) of an $n \times n$ chessboard, all spaces will be either occupied or threatened.

We know we can fill the main diagonal with queens, but what if we don't have n queens? Well, we can remove any queen and the cells she attacked are still covered. But can we remove two queens? Or three? According to Noam, we can remove any two queens with an even distance between them (i.e. an odd number of queens between them). If we pretend the queens are rooks, only four squares would fail to be threatened. However, the queen directly in the middle of the two queens (i.e. the "average" of the two queens that were removed) can threaten these squares through diagonal attacks, and we can use a parity argument to show that such a queen does not exist when the distance between the two queens is odd. Thus, the removed queens must be in all even or all odd positions.

Brian then announces that he is now a computer scientist, much to everyone's dismay. He indexes the n positions on the diagonal as $\{0, 1, 2, \ldots, n-1\}$. Without loss of generality, he removes queens from the even positions $2k_1, 2k_2, \ldots, 2k_\ell$ such that no element is the average of two other elements.

Next, Brian decides to make the chessboard infinite, and defines a subset $S \subseteq \mathbb{Z}_{\geq 0}$ to be *average-free* if no element of S is the average of 2 other elements of S. He then asks if it is possible to build an infinite average-free set. We begin constructing S greedily by going through each of the nonnegative integers in order and adding it to S only if adding it will maintain an average-free S. We observed that the elements of the set we constructed seemed to have something to do with powers of 3, so we tried writing out the numbers we added to S in base 3. Interestingly, all of the digits were 0s or 1s! Huh. Naturally, we conjectured that the set we constructed consisted of all non-negative integers whose base 3 representations contain only 0s and 1s.

We were presented with two questions. First, is this set actually what we get from our algorithm? That is, will the algorithm ever result in numbers outside the set? And second, is the set actually average-free? We proved the former by induction, inspired by Athithan's observation that if there existed an element k containing a digit 2 in base 3, we can find another element ℓ of S such that $\frac{k+\ell}{2} \in S$. And, we proved the latter by showing that the restriction on digits in base 2 forces $a = b$ if $\frac{a+b}{2} = c$ for $a, b, c \in S$.