
MATHILY 2021
WEEK 3
WEEK OF CHAOS

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Welcome…

… to the third issue of the 2021 RoM! This week, we had our Week of Chaos classes, where we thought about everything from surviving an apocalypse and saving the world (twice, using combinatorics and modeling) to origametry and analyzing a childhood best friend who is ~~imaginary~~ complex. Learn more by reading the Week 4 Calendar! In the Daily Gather Summaries, expect to read about keeping secrets from evil spy puffins, stupendous constructions and properties of syzygies, how an airport coloring is the same as a sudoku puzzle, assigning values to games to determine who will win, and the remarkable patterns of Fourier series (mod 7). Finally, there are many quotes, origami, poems, and apologies in the Fun Stuff section, along with an intriguing puzzle, The Mysterious Menagerie, and some Problems Recently Posed! Enjoy!

In this issue...

2.1 Kye: Apocalypse Planning (with Levity) BY Lijia and Ada

Imagine you are the mayor of Bryn Mawr, a place where the houses are organized into one line with the power source at the end. One night, the power goes out in our house. This means that there must be some house on the line between our house and the power source that is the source of the outage. We must find a way to discover the source of this power outage. But what is the optimal way to find this source? We were able to find a couple of strategies and count the expected number of houses we had to check for each one. In order to fix this power outage, we need to find the strongest person in town to help us. To do this, we arrange an arm wrestle tournament between the residents of the town, preferably in the most efficient way. We found a few algorithms that helped us sort the people according to strength and calculated the expected number of arm wrestles needed for each one. After solving these problems, we noticed that they all used a similar method of "divide and conquer", where we could split up the problem into smaller sections and solve those, then recombine. Now we need to find the shortest distance between two cities from a map of city locations (a list of (x, y) coordinates of cities). We found a couple of algorithms to find this distance and calculated the expected number of distances we had to measure for each algorithm. For the next problem, we are given a graph of cities connected by roads, with each road labelled with a length. We need to find the least distance from one city to another, travelling along the roads. How do we do this while also testing the least amount of roads? We found and proved an optimal algorithm (that didn't require testing all roads) and measured the expected number of roads we had to test using this algorithm.

2.2 Natasha and Patrick: Game Theory BY Kieran and Peter

Odd and Even Patrick and Natasha play a game in which each of them can choose either 1 or 2. If the sum is odd, Natasha pays the sum's value in money to Patrick, and if it is even, Patrick pays Natasha. As Patrick's and Natasha's consulting teams, we organized the outcomes into baos/trays and tried to find the optimal strategy for each player.

$$\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

We proved that the optimal strategy for both players occurred at the intersection point between the two lines bounding all possible outcomes, where neither player is able to change their strategy for superior results.

The Desecration of GatherTown sarah-marie has received an anonymous tip that GatherTown will soon go up in flames and brings Natasha and Patrick in for questioning. Realizing that they have been caught, Natasha and Patrick call upon their consulting teams to help them decide whether to snitch on each other or deny any knowledge during their interrogations. If both of them deny it, they will share 1 day of problem set critiques. If one of them snitches and the other denies, the one who denied will get 2 days of critiques, and the one who snitched will get none. Finally, if both of them snitch, they will share 3 days of critiques.

$$\begin{bmatrix} (-1.5, -1.5) & (0, -2) \\ (-2, 0) & (-0.5, -0.5) \end{bmatrix}$$

We found that if they can't communicate, the best strategy for both players is to snitch, since each value in the first row/column is preferable to the corresponding value in the other row/column. (We call rows/columns like these "bougie.") However, the strategy changes if other factors are in play, such as the ability to communicate or friendship between the players.

Following the advice of their consulting teams, at the Volcano Drill, Patrick immediately snitches when questioned about the desecration of GatherTown.

2.3 Brian: What norms can do for you BY Vidur and Jeffrey

How do we make the real numbers out of the rationals? We set out of the gates with this question in mind, and decided that we could construct the real numbers by taking the set of sequences of rational numbers that *should converge* (get arbitrarily close) to some real number - that real number being called the *limit* of the sequence. We also noted that, specifically, using the decimal expansion of a number could help us attain all these real numbers. But this idea of "converging" is based on a specific notion of size - absolute value - and what it means to be "close to" something as well. So, we defined a *norm* as a measure of size of some number x - written as $|||x|||$ - with three properties. We focused on a specific norm, based around an arbitrary prime p, denoted $|||x|||_p$. Using this new measure of size changed a lot - sequence that converged with our standard measure of size were often no longer converging and many sequences that didn't converge now converged (for instance, this new measure of size allowed us to correctly say that $\sum_{i=0}^{\infty} 2^i$ converged to -1, for $p = 2$). We continued to explore this new world, discovering fascinating properties and new ways to express numbers like i and $\sqrt{2}$. Who knew that size could change so much?

2.7 Nadav: Information Theory! BY Jason

How do you create an optimal code to send English text in the least amount of bits? Are you sure that your code is understandable? What if you have bad WiFi? How do you measure information? A code is a mapping of events to words. Specifically, we considered English letters mapped to binary strings (some sequence of 1s and 0s). However, 1s and 0s can be ambiguous, so for our code to be valid, it must not be ambiguous. To do this, we established the Bony Web Property, which states that no word can appear as the prefix of another word. Similarly, the Reverse Bony Web Property is the reverse, where no word appears as the suffix of another. Either property is sufficient for a code to be valid. What do we mean by an optimal code? We introduced the idea of expected length, the expected value of the length of a word in a code. Note that the probability of a word occurring depends on the frequency distributions of letters (not all letters are equally likely). An optimal code has the least expected length. We proved that a greedy approach to minimizing expected length using a binary tree is optimal (if interested, research huffman coding). We then considered some codes that have error detecting and error correcting capabilities (for when bits get flipped, perhaps due to bad WiFi). Some ideas include introducing parity bits or simply sending the message multiple times. We also came up with a notion of quantifying information (entropy), using the idea that information should be additive and less probable events reveal more information.

2.8 sarah-marie: Math Saves the World: Combinatorial Optimization BY Hana

WARNING: This class contains no knowledge of how to save the world. Only how to make things better or, perhaps more fittingly, optimal. In this class we learned how to solve combinatorial optimization problems. First, set up a problem by writing the situation in terms of variables (binary or sum of binary variables that represent the things that can be changed) and constraints (linear inequalities that represent our limits). For something like travelling to the moon, variables might be how much food, cameras, and oxygen we are bringing and our constraints may be a weight limit and a budget. After this, we write an objective equation which we can either maximize or minimize and use it to solve. How do we solve, I hear you asking? One method is the Simplex Method, which is more reliant on boxes/baos/trays and another is the Branch-and-Bound method which uses a divide and conquer method to solve. Both can be used to solve optimization rather quickly for smaller problems (if you discount the time it takes to input the numbers into a computer), but larger problems can take up to months to solve. There are many reasons to use combinatorial optimization in the real world. Optimization can be used to find everything from optimal fire station distribution to human body temperature.

2.14 Corrine: Erdős Magic BY Jennifer (and 8 percent of Nishka)

War! We started this class by settling a war between two factions of Umlaut Snakes. How? Probability, expectation, and a touch of magic. We proved when there could exist a coloring on n amps without a k-tuple of Doom.

A general way to structure these types of (Erdős) magic proofs:

1. Start with a question about existence (i.e., Do I exist?)
2. Inject probability
3. Define state space, probability function, etc.
4. Use probability or linearity of expectation to prove that something exists
5. Be happy that your existential crisis was averted!

And Peace We practiced implementing these techniques with other problems. Specifically, we looked at properties of two special AV systems, RAVs and DAVs. We found the expected number of wires, k tuples of doom, and loops on a RAV with n-amps in addition to finding when a DAV system would have the winning property and when no sub-DAVs would be transitive. We then looked at the relationship between expectation and probability and found that, in general, if we know $E[X]$, we can bound $P(X \geq b)$ for some $b \in \mathbb{R}_{\geq 0}$. Finally, we pulled together everything we had learned, used asymptotics by waving our wand a bit, and proved a cool property about the relationship between Peter's number (see: artistic anthropods) and the shoop (**sh**ortest l**oop**) number of an AV system.

At last, both Umlaut snake factions live in harmony.

2.15 Brian: Möbius Transformations Revealed BY Ada

As seen during the Daily Gather...Mobius Transformations! Working with complex numbers, we found the functions for the four "building blocks": translations, rotations, dilations, and (new!) inversions. We combined these building blocks to find the general form for Mobius Transformations and confirm that it is the same as the one we saw in the movie. With this general form, we found an inverse and its domain. Up until this point, we had been working in a 2D coordinate plane. So we added a z-axis and moved into a 3D coordinate system. We added a sphere and its light and realized that it was a projection. Then we found that the north pole on the sphere is sent to infinity and vice versa, and explored what it meant to project a circle onto the plane. Then we proved the video's mention that right angles stay true during projections and found a system for measuring angles between lines and circles to make this easier for us. In fact, we proved that all angles stay true during projections.

2.27 Daniel: Too Much Slack Syndrome BY Oliver

A Dangerous Affliction In a world where all in-person contact is forbidden, a frightening new disease has surfaced that threatens all of society. Patient Zero Sophie has been diagnosed with the first case of Too Much Slack Syndrome. Patients with this disease are forced to bounce back and forth between Slack channels...forever! For example, Sophie was forced to bounce around the pets, chit-chat, and fun-links channels. Every minute, Sophie had a certain probability of either staying in her channel or moving to either of the other two. In order to study this peculiar affliction, we developed the TMS Bao method of interpreting this movement. Each column of our Bao would represent the channel Sophie was currently in, and each row would represent the probability of staying or moving into another channel, based on which channel she was currently in. Since Sophie was locked in Slack forever, the probabilities in each column would sum to one. Then, we could use Bao multiplication to compare the probability of Sophie going to each chamber, given a starting chamber. Kiran proposed that for all TMS Baos, the TMS patient would eventually settle into a consistent probability of being in each chamber. However, we recognized multiple counter examples to this claim. Then, we proposed the JOSK conjecture, which stated that Kiran's proposed limit would exist as long as none of the entries in the TMS Bao were zero. In order to prove the JOSK conjecture for a 2×2 TMS Bao we showed that the entries in each row would approach each other, and that they shared a common limit.

2.28 Josh and Johnna: Somewhat Dirty Algebra BY Owen

While things like muffins, baos, trays, delis, bakeries, and PDLs were nothing new to us thanks to Root, in "Dirty" Algebra we pushed these discoveries as much as possible. After everyone began to argue about which Root class had the superior naming schemes, we decided to standardize our definitions by introducing terms like "Zomboli Tuppers" and "Munchkin Zombolis". We explored morphisms between two zomboli tuppers and how to find the PDL or donut hole, and finally explored the size of a munchkin zomboli. In the end, we were able to prove that the size of any munchkin zomboli divides the size of the zomboli with a combinatorial proof, and we had parting thoughts on how we could use this fact to prove a seemingly "gross" divisibility condition, and the existence of the "order" of elements in a zomboli.