3.1 Emi - Bees or Something BY Charlie

We started Daily Gather by considering bees. In particular, we wanted to think about a bee named Barry who lives in 2 dimensional space, has a point body and mile long stinger, and therefore has no 2-D area. Emi informed us that bees communicate using dances. The first dance we considered was the Flower Dance. The rules of the Flower dance were that Barry must dance continuously (he can't teleport) and that he must move from the line he is on to a parallel line. Being a polite bee, Barry would like to minimize the area he sweeps out when doing his dances. We found that we could create a Flower Dance of arbitrarily small length by rotating Barry by a tiny angle, shifting Barry to the line he wants to go to, and then rotating back by the tiny angle. We then considered a new continuous dance called the Hive Dance. The Hive dance requires that Barry ends the dance in his starting position and that he points in every possible direction along the way. Emi left us to answer the question of what some efficient dance paths might be? (Efficient means taking up the least amount of area.)

A general idea that most groups played with was maximizing the overlap of the areas swept out by Barry's movements in order to minimize the area. For example, several groups considered making a sort of 'puffy triangle' by rotating Barry along the sides of an equilateral triangle with side lengths that are the size of Barry. Other groups looked at combining rotations with translations to produce curves of potentially even smaller area. One process that Emi had us focus on was rotating about a triangle with a height equal to the length of Barry. Emi suggested that we continue to apply our idea of maximizing the overlap of areas, and showed us that by cutting the triangle in half and placing them on top of each other, a more efficient dance might be possible.

But is this a valid dance path? We found that it wasn't, because Barry only turned through 60°, but Emi asked if it was possible for us to achieve this 60° dance with even less area. We decided that we could, just by slicing the triangle up some more! We tried splitting the triangle into thirds and placing the pieces on top of each other. We then tried to make sure there would always be a valid path for Barry to dance around the whole 60°, and argued that since Barry could complete his dance on the whole non-cut triangle and we were just translating pieces of that triangle horizontally, all Barry has to do is move horizontally as well to complete his dance. So what happens if we keep cutting? We decided that we can make the area swept out by Barry's 60° Hive Dance arbitrarily small by cutting the triangle into more and more overlapping pieces! This is nice and all, but a little unsatisfying since Barry isn't actually doing a Hive Dance, he's only going through 60°. So we constructed a generalized path for Barry to follow which involved placing cut-up triangle hats around a circle and rotating through each triangular section by sweeping through and sliding between sections. We found that rotating through this structure twice would have Barry point in all 360° and would bring him back to where he started. And by cutting the original triangle into tinier slices, we decided that Barry's Hive Dance could be as small as we want it to be(e)!

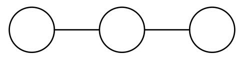
Emi ended her Daily Gather by reminding us that mathematics and human-people interactions don't have to be separate and that it is in general a good idea to be human. Then someone pasted the Bee Movie script on limnu and we were instructed to leave and marinate in teleporting bees.

3.2 MathILy-EST—If you give a mouse a cheese ... BY Benjamin G

The MathILy-EST students used their 90 minutes of Daily Gather time to talk about trapping mice in rooms, and instructing them with radio broadcasts to move throught small tunnels back to their depressing homes. Mouse configurations consist of a series of rooms connected by tunnels. A broadcast tells the mice in any adjacent pair of rooms to swap. We said that the Individual Broadcast Number of a mouse configuration M, IBN(M), is equal to k if

1. $\exists$ a configuration of mice that takes at least k broadcasts to send all the mice back to their homes.
2. All configurations can be routed home in no more than k broadcasts.

To get a hang of it, we tried to find the IBN of the following configuration:



We concluded that this had an IBN of 3, but more importantly, some groups discovered a generalized way to find the IBN of a configuration:

1. Create an EAI with an HSF for each mouse arrangement.
2. Connect two configurations iff they are one broadcast away from each other.
3. Find the IBN of this configuration by finding the longest path between any two arrangements.

But, for a more complicated configuration, this is too hard, so instead we talked about sorting mice by society instead of individually. Each mouse belongs to the blue or orange society, and each home in the configuration is either orange or blue. At the end of the broadcasts, each mouse wants to be in any home of their color. We define SBN(M) similarly to IBN.

Now we were given the following theorem: $\text{SBN}(M(w, h)) \leq 2(w + h)$, where M is a rectangular mouse maze with width w and height h. Using this, we want to find an upper bound for $\text{IBN}(M(w, h))$.

After talking in breakout rooms, we came up with a recursive algorithm to use societal broadcasting recursively, in order to sort individually. The algorithm is as follows:

1. Color the top half of the homes blue, and the bottom half orange. Color the mice who live in the top half blue, and the bottom half orange. It takes at most $2(w + h)$ broadcasts to sort these mice into their halves.
2. Repeat (1) with a vertical divider. This also takes at most $2(w + h)$ broadcasts. Note that (1) + (2) takes at most $4(w + h$ broadcasts in total.
3. Repeat (1) and (2) within the quadrants created. Because the width and height are both halved, this only takes $4\left(\frac{1}{2}w + \frac{1}{2}h\right)$ broadcasts to do.

The next time we repeat it will take $4\left(\frac{1}{4}w + \frac{1}{4}h\right)$, and so on. Eventually, the quadrants will only have one mouse inside, so no more divisions will be needed. Thus, $\text{IBN}(M(w, h)) \leq 4(w + h) + 4\left(\frac{1}{2}w + \frac{1}{2}h\right) + 4\left(\frac{1}{4}w + \frac{1}{4}h\right) + \cdots \leq 8(w + h)$.

3.3 Math Movies! BY Jiakang and Kishore

Our first movie would be *The Divided Man: Committment or Compromise*, The sad story of a man trying to find his better half. He is forced to make a decision at a crossroads, but unable to make a choice, he splits in half and goes along both roads. But there's a happy ending as the divided man joins together at the end! The music was great as usual with an especially DISEASED bass line. We then watched *Fronts and Centers* and saw how 3D shapes are formed via curved lines spinning and spinning and spinning. It was very trippy and psychedelic, and of course: egg. We then watched *Canon*, another movie with great music, this time the classic Frere Jacques, as we watched both toy cubes, hands, and even people, dance in real life, with a cameo from some cats! The next movie would be *Geodesics and Waves*, where we learned that geodesic curves are the shortest curve between two points and can be considered the straightest curves on a surface. For example, on a globe, geodesics are arcs on great circles. Also we saw that when considering geodesics on a cube, they became discontinuous at vertices. Continuing the trend of geometry related math movies, we turned to *Möbius Transformations*, where we learned that the 4 basic Möbius transformations are translations, rotations, dilations, and inversions. We can understand how these work by considering a sphere above the plane and shining a light from above the sphere and seeing the projection. Moving the light or the sphere would correspond to a Möbius transformation: translating the sphere would translate the projection, rotating the sphere along a vertical axis would rotate the projection, moving the light up/down would dilate the projection, and rotating the sphere along a horizontal axis would invert the projection. Then we saw the bizarre movie, *Some adventures of Rhombus Worms*, where we witnessed rhombus worms being curled to form rosettes! Then came the very relatable *Attack of the Note Sheep*, where someone's sheep doodles have risen up and started attacking their notes about quaternions! In two videos titled *Infinite Series*, we first learned about loops and how to multiply them, and it turns out that loop concatenation is not associative. We then learned about homotopies and their related associahedra. We then watched a frustrating video about the efficiency of airplane boarding methods. Apparently, back to front boarding groups are really bad, even worse than literally doing nothing. The best way to board is the Steffen perfect, which is back to front, windows in, with alternating rows. Unfortunately, this is impossible to implement with humans because lots of boarding groups. We ended our last ever math movies session with a blast from the past, *Dance Squared*. We get to see once again the squares a dancing and the fiddle a playing. With that, we finished the last math movie session of MathILy 2020.

3.4 Nate Harman - How many even numbers are in the millionth row of Pascal's triangle? BY Jonah

Nate Harman from the Institute of Advanced Study is a MathILy-Er Lead Instructor and is interested in the intersection of representation theory with other areas of math. He wrote a paper about the card game SET and another about integer inverses of matrices. One of Nate's earliest experiences with math was trying out for his high school's math team to impress a girl. Instead, he impressed his teachers and continues to impress us all. Thanks for the Daily Gather presentation, and I hope that some day you too can join Mark Watney in space.

Gather At the beginning of our meeting, Nate started us off with drawing Pascal's triangle and then circling even numbers. Although some errors were made in our collective calculations, our smaller-row elements were mostly fine. It appeared that the circled even numbers formed Sierpinski's triangle! However, we were not sure how we could either capture this observation mathematically or use it to answer our primary question: how many even numbers are there in the nth row of Pascal's triangle? After discussing in breakout rooms, we reconvened to share our thoughts. Pablo observed that if we consider diagonals (also called columns) of the triangle, which have values $\binom{n}{k}$ for some fixed whole k and $n \geq k$ for all $n \in \mathbb{W}$, then, if we consider the sequence $S = \binom{k}{k}, \binom{k+1}{k}, \ldots$, k even means that S will alternate between k odds and k evens while k odd means that S will alternate between 1 odd and k evens. Annabel noted that rows of power 2 have all evens except for their first and last elements while rows of one less than a power of 2 have all odd elements. Jiakang claimed that Pascal's triangle with $2^j - 1$ rows for $j \in \mathbb{W}$ is comprised of three identical $2^{j-1} - 1$ triangles and a same-size triangle of zeros that borders each of the $2^{j-1} - 1$ triangles. Easton shared that if we list out the number of evens in the nth row as a sequence P_n, then elements 2^j through $2^j + 2^{j-1}$ for $j \in \mathbb{N}$ are equal to corresponding elements in P from 2^{j-1} to 2^j plus 2^{j-1}. He gave the example: $P_4 = 3, P_5 = 2, P_6 = 3, P_7 = 0 \implies P_8 = 7, P_9 = 6, P_{10} = 7, P_{11} = 4$. Although helpful, this form does not give us the remaining elements from $2^j + 2^{j-1} + 1$ through $2^{j+1} - 1$. Related to and expanding on Easton's ideas were Sejal's, one of which was a conjecture that the number of odds in a row n is 2^p, where p is the number of 1s in the binary representation of n.

We once again separated into breakout groups to discuss possible proofs of the previously mentioned conjectures and observations. Powerful ideas emerged. Ben K.'s group had two ideas, one for inductively proving that the 2^jth row has all even elements except for the first and last elements and one for modular congruence. The first claim was equivalent to $(x + y)^{2^j}$ has all even coefficients except for the x^{2^j} and y^{2^j} terms. The base case of the induction is true, namely $x + y$, so let our inductive hypothesis be that $(x + y)^{2^{p-1}}$ satisfies the desired condition for some whole $p << j$. Then, we want $(x + y)^{2^p}$ to satisfy the desired condition assuming our inductive hypothesis.

$$(x + y)^{2^p} = (x + y)^{2^{p-1}}(x + y)^{2^{p-1}},$$

so the only possible odd term (besides first and last) that could exist in the expansion of $(x+y)^{2^p}$ is $x^{2^{p-1}}y^{2^{p-1}}$. However, there are two possibilities for the formation of this term, namely taking the $x^{2^{p-1}}$ from the first $(x + y)^{2^{p-1}}$ and $y^{2^{p-1}}$ from the other, and vice versa, so that the $x^{2^{p-1}}y^{2^{p-1}}$ term has coefficient 2 and thus the induction step is complete. The second idea from this group was

$$\binom{n}{k} \equiv \binom{n + 2^j}{k} \pmod{2}$$

for $2^j > n$. By Vandermonde's identity, which can be proven by a combinatorial argument, and the first idea, which is equivalent to

$$\binom{2^j}{r} \equiv 0 \pmod 2$$

for all natural $r < 2^j$,

$$\binom{n+2^j}{k} \equiv \sum_{i=0}^{k} \binom{n}{i}\binom{2^j}{k-i} \equiv \binom{n}{k} \pmod 2.$$

As a result, $\binom{n}{k} \equiv \binom{n}{n-k} \equiv \binom{n+2^j}{n-k} \equiv \binom{n+2^j}{k+2^j} \mod 2$. Combining these ideas, we can express an idea Will had, which was to find and compare digits in the binary representations of n and k. We can reduce $\binom{n}{k} \mod 2$ repeatedly by deleting the leading digit of both k and n iff $(n_{\text{leading digit}}, k_{\text{leading digit}}) \in \{(0,0), (1,0), (1,1)\}$. But if at any point $(n_{\text{leading digit}}, k_{\text{leading digit}}) = (0,1)$, then k can be reduced by our identities to be larger than n so that $\binom{n}{k} = 0$. Therefore, the only way that $\binom{n}{k} \equiv 1 \mod 2$ is if for each 0 in the pth place of the binary representation of n, the pth place of the binary representation of k must be 0. Since we don't care about the case where the pth place is 1, to count all k such that $\binom{n}{k} \equiv 1 \mod 2$ for a fixed n, we have two choices for every 1 in the binary representation of n and one for every 0. Will's argument illustrates why Sejal's conjecture is true! And, knowing Sejal's conjecture/theorem/thing-we-now-know-is-true, we can count evens by subtracting odds from the number of elements in a row n, $n+1$.

We can rest easy tonight knowing that the number of even numbers in the millionth row of Pascal's triangle is, given $1,000,000 = 11110100001001000000_2$, $1,000,001 - 2^7 = \boxed{999,873}$. Zzzzz... Wait, what? You mean we can do something similar for $\mod p, p \in \mathbb{N}$? Oh no, I'll never get ANY sleep!