2 Daily Gathers

2.1 Monday: Sets without SETs

by Ian S

sarah-marie opened her paradoxically-titled Daily Gather with an explanation of how the card game Set is played. Each of the 81 cards has 4 attributes: number (1, 2 or 3), shape (squiggle, oval, or diamond), filling (solid, open, or dashed), and color (red, green, or purple (or in this case, grey, gray, and græy)).

A set consists of 3 cards that satisfy the following criterion: for each of the 4 attributes, the three cards are either all the same or all different. For example, in Figure 7, the diagonal from the top left to bottom right forms a set, because all three are red, all three are ovals, all have different numbers, and all have different fillings. However, the center, bottom left, and bottom middle do not form a set, because two of these three cards have 2 shapes, and one has 1 shape.

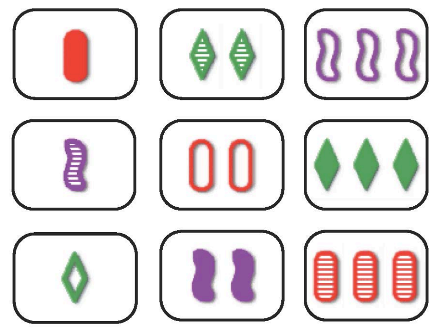

Figure 7: An example of Set cards.

sarah-marie then asked what the maximum number of cards is that contains no 3 that form a set. Students quickly noticed that a collection of 20 cards was posted on the front wall among which there were seemingly no sets, which caused us to conjecture that this was the maximum number of cards with this property.

A single card can be modeled by an ordered 4-tuple of elements of $\mathbb{Z}_3$. Geometrically, we saw that the condition for three cards forming a set is equivalent to their three points being collinear in 4-space. The goal was now to find the maximum number of points in a $3 \times 3 \times 3 \times 3$ cube with no three on a line. This problem did not seem very approachable, so we began with lower-dimensional versions.

In the fascinating game of one-attribute set, there are a total of three cards, which form a set. So, the maximum number of cards that contain no set is 2. When we considered two-attribute set, we saw relatively quickly that any 5 of the 9 cards form a set, implying a maximum of 4. After some thought, we found a general method for finding the maximum size of a set-free set with some number of attributes, which we used to verify the claim that 20 was indeed the maximum size of a set that contains no sets in the conventional game.

In the final minutes of her daily gather, sarah-marie discussed some recent research relevant on this problem. In 2016, an upper bound for the maximum size of a set-free set with n attributes was proven to be $O(2.756^n)$. To everyone's amazement, our very own Nate had recently published a paper on the number of k-element set-free sets in an n-attribute deck, in which he showed that the answer was an polynomial in n with integer coefficients that alternating signs.

2.2 Tuesday: Math Movies

by Hanna

The thrilling conclusion to Gideon's daily gather

By transforming the problem of rows and columns summing to odd or even numbers into a problem where the products of rows and columns turn out to either be positive or negative, and using the cell that Frank filled with lighting bolts and Xs and Ys and Zs, you and your talking dog (with the help of some qubits) were able to devise a perfect strategy, so that you will avoid being jailed.

3Blue1Brown

We found out about the practical uses of topology in solving the inscribed rectangle problem, and learned about how a torus can represent ordered pair on a loop, or how Möbius strips can represent unordered pairs of points on a loop.

Flatland

We followed the story of a square in a world where women are literal line segments, while men are all sorts of polygons, ordered by the number of sides they have [Editor's note: this is sexist]. A square has a grandson, a small hexagon, who wonders about what a moving square forms. The square is skeptical, until he meets a sphere and is able to see from the third dimension. However, when he returns from the third dimension, no one else in the town believes him, and he is imprisoned for a large amount of years, and he is never able to see the third dimensions again.

Hypercube Projections and Slicing

We saw how cubes could be formed out of moving squares, and tried to generalize this by putting together two moving cubes to create a 4-cube. Other ways to visualize this is through perspective drawing or slices of the 4-cube.

Dihedral Kaleidoscopes

Kaleidoscopes can be used for a variety of different things, including tesselations. We can create one by placing mirrors at an angle. Mirrors reverse the sense of an object – a right hand will become a left hand in the mirror world. Special values include $2\pi/n$, which will give n-fold symmetry.

2.3 Wednesday: Finding Sets of Friendly Numbers

by Abhinav

Bill Martin is a professor at Worcester Polytechnic Institute. His talk on Wednesday began with a discussion of characteristics of Toucaets, a kind of imaginary bird. Values are assigned to variables (like flightless or flying), each of which has two possible values, in order to satisfy some system of constraints on pairs of variables.

The various characteristics of and conditions on Toucaets were considered, so for example if a Toucaet had to satisfy being either one of blue-feathered or round-beaked, and either one of pointy-beaked or unwebbed feet, and either one of webbed feet or blue-feathered, then it can have blue feathers and a pointed beak, which satisfied all the necessary conditions. This idea can be made abstract, so if characteristics are labeled X_i, and the relations between pairs of them are represented using Boolean Algebra, then we can re-write the above conditions as follows:

1. Blue Feathers = X_1
2. Pointed Beak = X_2
3. Webbed Feet = X_3
4. And = $\wedge$, Or = $\wedge$, Not X = $\overline{X}$
5. System $\rightarrow (X_1 \ \vee \ \overline{X_2}) \wedge (X_2 \ \vee \ \overline{X_3}) \wedge (X_1 \vee X_3)$

This is relatively easy to solve by hand, but when there are several million pairs of thousands of variables, then the use of computers comes in. Bill Martin went on to explain the differences between classical computers and quantum computers in this regard. Quantum computers have an edge, but there currently exist no true quantum computers. The D-Wave computer simply takes in inputs as 2-SAT problems from a normal CPU, and spits out a very good approximation of the best possible answer.

2.4 Thursday: I Demand Offerings, by Ba'al Josh

by Corwin

At the beginning, Josh gave an important life lesson: Success in mathematics does not depend on contests. He then listed 5 students, 5 crouges, and who could study what. Each student has some amount of blood, and each crouge requires a *sacrifice* of a certain amount of blood to be studied. Studying a crouge that requires at least as much blood as the student has will kill the student. We know that the amounts of blood the students have sum to strictly greater than the sum of the blood costs. The question is: For which arrangements *must* there exist a student-crouge pair such that the student can study the crouge without dying? We concluded that, under the assumption that the number of students is equal to the number of crouges, such a student-crouge pair (a *blood sacrifice*) always exists iff every set S of students can collectively study at least $|S|$ students.

Another blood problem: Suppose we draw a sea creature of students and friendships. They have been trading blood, and blood debt exists. Hence, some people have negative amounts of blood, but the sum of everyone's amount of blood is positive. This time, you can sacrifice a friendship if the endjoints sum to a positive amount of blood. Not all sea creatures guarantee that a sacrifice is possible, as shown in Figure 8.

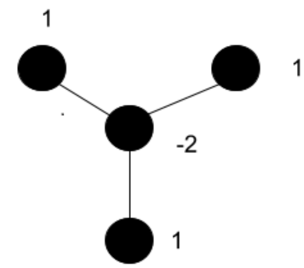

Figure 8: Some configurations of blood with this network of friendships allow no friendships to be sacrificed.

We considered which sea creatures do guarantee a sacrifice. A sufficient condition is that the sea creature can be partitioned into cycles and pairs of adjacent joints. However, it turns out that a sea creature can be converted to the crouge-studying model, where the crouges are versions of the students, but their blood amount is the negation of the normal version of the student. A blood sacrifice exists if and only if a student can study a crouge. Josh also invited us to show later that a partitioning into cycles and adjacent joints is a necessary condition for a guaranteed blood sacrifice.