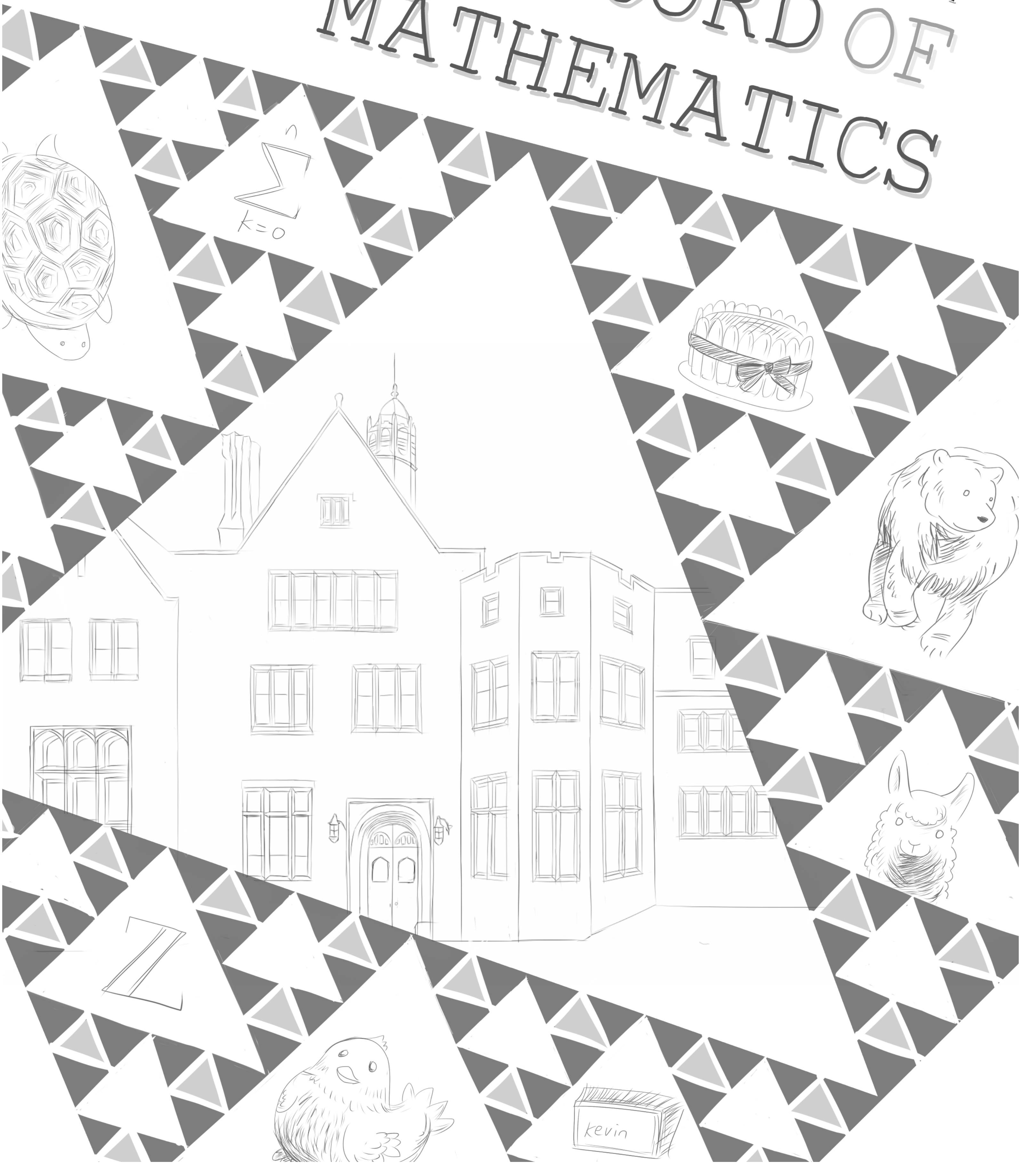
2014

WEEK 4

RECORD OF MATHEMATICS



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Acknowledgements

Senior Editors: Erica and Natasha
Juñor Editors: Mo
Instructors: sarah-marie, Tom, Hannah, Nate
PRiME: Corrine

Calendar for Week 2 of Branch Classes at MathILy (Week 5 overall):

A reminder of our weekday-ly schedule, except for Friday this week:

Breakfast 8:15–8:55 at Erdman
Morning classes 9:00–1:00 in Park 336 and 337
Lunch 1:05–1:30 at Erdman
Daily Gather 4:30–5:30 in Park 336
Dinner 5:35–6:30 at Erdman
Evening classes 7:00–10:00, Park 336 and 337

Special things:

- This is our last week. Think early about packing your stuff (or at least about cleaning your room).
- Make sure Corrine has your travel information.
- On Wednesday, receive instructions on reflection/introspection and on Thursday before the start of evening class, send completed introspections to your instructors.
- On Thursday, at 3:30: Let's clean and tidy the Back Smoker!
- All non-last-minute Final ROM articles must be submitted for review by Thursday afternoon; the final submission deadline for review of last-minute Final ROM articles is 2:15 Friday.
- Meal hours on Saturday are the same as on other days.

Daily Gathers:

Monday: Kat Shultis (University of Nebraska)—Polynomial Genocide

We'll explore which polynomials are multiples of some nice polynomials and what happens when you kill off lots of them.

Tuesday: Rachel Shorey (Strategic Telemetry in DC)—From Math to President!

An enormous amount of math and computer science goes into a modern US presidential campaign. But it's not as easy as doing a few proofs and then knocking on the door of the White House. I'll talk a bit about the theory behind these models, a lot about making them work with messy, real-world data, and a bit about how much fun it is to be a resident nerd on a political campaign.

Wednesday: Ethan Berkove (Lafayette College)—Flexagons: a "simple" mathematician's toy

Flexagons are geometric shapes that can be flexed to show different faces. The simplest are easy to make, but flexagons come in all shapes, sizes, and levels of complexity. We'll have a chance to fold some flexagons, and describe some ways to analyze their behavior.

Thursday: Hannah—Laser-eye Slugs Amid Leaves of Lettuce

Here's what it's *really* like to be at an REU.

Friday schedule:

Breakfast 8:15–8:55 at Erdman
Morning classes 9:00–12:15 in Park 336 and Park 337
Tidy classrooms, group pictures, etc. 12:15–1:00
Lunch 1:05–1:30 at Erdman
Afternoon: everyone make sure they're packed to leave in the morning
Final interviews start around 3:00 in Denbigh
Dinner 5:35–6:30 at Erdman
Closing meeting in the Cloisters, 7:00
Starting around 9:30, a puzzle/game party in the Back Smoker

Saturday schedule:

Goodbye.

Emergency Triangles

Feel a need for triangles and only have n-gons for $n > 3$? Use the Equidecompsability theorem to manufacture whatever triangles you desire. No need to call, the triangles are already at your fingertips.

Convex Polytopes - Week 1

By: Gideon

Linear, Affine, Convex

A set $S \in R^d$ is convex if $\forall x, y \in S, \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in S$. Equivalently, a convex set contains all line segments between its elements. We defined the terms "linear", "affine", and "convex" in various contexts: A linear combination of elements of a set S is of the form $\Sigma c_i s_i$, for $c_i \in R$, an affine combination is a linear combination such that $\Sigma c_i = 1$, and a convex combination is an affine combination such that each $c_i \in [0, 1]$. For each of these, the LAC (linear, affine, convex) hull of a set S is the set of all LAC combinations of the elements of S, a LAC subspace is a set that contains all LAC combinations of its elements, and a LA function preserves LA combinations. We proved that a LAC hull is a LAC subspace, and defined the dimension of a convex set S to be the dimension of $aff(S)$.

(In)dependence

A set S is LAC-ly independent if no element can be expressed as a LAC combination of other elements. S is LA-ly dependent if some element can be expressed as a LA combination of other elements, but only elements can be convexly dependent, not sets.

Specific Objects

We discussed d-cubes, which we first defined pictorially, then recursively, as $C_d = C_{d-1} \times [0, 1]$, from which it follows that $C_d = [0, 1]^d$. In order to better quantify cubes, we defined $f_k(C_d)$ as the number of k-dimensional faces (not yet defined) in the d-cube, and found the formulas $f_k(C_d) = \binom{d}{k}2^{d-k}$, $f_k(C_d) = 2f_k(C_{d-1}) + f_{k-1}(C_{d-1})$. We extended notions of triangles and tetrahedra to simplices: a d-simplex is $conv(S)$ for some affinely independent set S with $|S| = d + 1$. We found the closed formula $f_k(\Delta_d) = \binom{d+1}{k+1}$, and the recursive formula $f_k(\Delta_d) = f_k(\Delta_{d-1}) + f_{k-1}(\Delta_{d-1})$. We extended squares, octahedra, etc. to crosspolytopes: A d-crosspolytope is $conv(\{e_1, -e_1, e_2, -e_2, ..., e_d, -e_d\})$. We found a closed formula: $f_k(X_d) = \binom{d}{k+1}2^{k+1}$, unless $k = d$, where $f_d(X_d) = 1$. All faces of a crosspolytope are simplices.

Four Undefined Terms

For a (convex) set C, we used the terms $int(C), \partial(C), relint(C), rel\partial(C)$ to discuss qualities of polytopes. $int(C)$ means the interior of C, $\partial(C)$ means the boundary of C (both in the "ambient" space of C), while $relint(C)$ means the relative interior of C, and $rel\partial(C)$ means the relative boundary of C (with respect to the subspace containing C).

Polytopes

An extremal point (a vertex) of a convex set C is a point x such that $C\backslash\{x\}$ is convex. The set of extremal points of a convex set C is $ext(C)$. We wanted a polytope to be convex, bounded, have boundaries in all directions, and have a finite number of extremal points. This led to the definition: a d-polytope P is $conv(S)$ for a finite set S such that $conv(S)$ has d dimensions. We also proved that for a polytope P, $conv(ext(P)) = P$. It was shown that not all convex sets have convex bases, but all polytopes have a convex basis, and only simplices have unique coordinates in such a basis.

General Extensions of Objects

A pyramid over a polytope P is $pyr(P) = conv(P \cup \{x\})$ for some x such that x, S are affinely independent. This satisfies $f_k(pyr(P)) = f_k(P) + f_{k-1}(P)$, and thus generalizes simplices (going up one dimension). A prism over a polytope P is $pzm(P) = P \times [0, 1]$. This satisfies $f_k(pzm(P)) = 2f_k(P) + f_{k-1}(P)$, and thus generalizes cubes. A bipyramid over a polytope P is $conv(P \cup \{x_1, x_2\})$, where x_1, x_2 are affinely independent of P, and there exists $x \in relint(P)$ such that $x_2 = 2x - x_1$. This generalizes crosspolytopes.

Hyperplanes and Khalfspaces

We defined $H(\vec{y}, \alpha) = \{\vec{x} \in R^d | \vec{x} \cdot \vec{y} = \alpha\}$ (a convex $(d-1)$-flat in R^d), and $K(\vec{y}, \alpha) = \{\vec{x} \in R^d | \vec{x} \cdot \vec{y} \leq \alpha\}$ (a convex (k)halfspace in R^d). For practical purposes, we generally reduce all hyperplanes and khalfspaces so they are of the form $H(\vec{y}, 1)$, $K(\vec{y}, 1)$. We also began to prove that $\bigcap_i K_i = K \Leftrightarrow y \in conv(\{y_1, ..., y_n\})$, where each K_i is associated with y_i.

Faces

We gradually created several preliminary (but soon to be proved equivalent (hopefully)) definitions for faces:

1. A face of a polytope P is $H(y, 1) \cap P$ such that $P \in K(y, 1)$.
2. A face F is $conv(A)$ for some $A \subseteq ext(P)$ such that $F \subseteq rel\partial(P)$ or $F = P$, and $\forall x \in ext(P) \backslash A,\ dim(conv(A \cup \{x\})) > dim(F)$.
3. A face $F \subseteq P$ is a convex set such that $P \backslash F$ is convex, and $\forall x \in F, y \in P \backslash F, (x, y] \in P \backslash F$.
4. A face $F \subseteq P$ is a convex set such that $(x, y) \cap F$ is nonempty $\Rightarrow x, y \in F$.

Face Lattice

A poset (partially ordered set) is a set S with a relation $\leq$ such that $\forall a, b, c \in S$, $a \leq a$ (reflexive), $a \leq b$ and $b \leq a \Rightarrow a = b$ (symmetric), $a \leq b$ and $b \leq c \Rightarrow a \leq c$. The face lattice $L(P)$ of a polytope P is the poset of faces ordered by inclusion.

Equivalence

Two polytopes P_1, P_2 are affinely isomorphic/equivalent if there exists an affine bijection $f : P_1 \to P_2$. They are combinatorially isomorphic/equivalent if there exists a bijection $\varphi : L(P_1) \to L(P_2)$ such that $\phi(a) \subseteq \phi(b) \Leftrightarrow a \subseteq b$. We established that the f-vector of a polytope does not determine the polytope up to combinatorial equivalence by finding two inequivalent polytopes with the same f-vectors.

Duals

We noticed that the f-vector (the vector $(f_{-1}(P), f_0(P), f_1(P), \ldots, f_d(P))$) of the d-crosspolytope is the reverse of the f-vector of the d-cube, and that the f-vector of the d-simplex is its own reverse. This was revealed to have some possible geometric motivation when we created an octahedron from the midpoints of the sides of a cube, and created a cube from the midpoints of the sides of an octahedron. We say that two polytopes P_1, P_2 are combinatorial duals if there exists some bijection between their face lattices that reverses inclusion, i.e. a function $\phi : L(P_1) \to L(P_2)$ such that $\phi(a) \subseteq \phi(b) \Leftrightarrow a \supseteq b$. We conjectured (and mostly proved) that each polytope has a combinatorial dual, that any two duals of the same polytope are combinatorially equivalent, that for any polytope P, $pzm(P)$ and $bip(P)$ are combinatorial duals, and $pyr(P)$ is its own dual.

Epotylops

An Epotylop is a bounded intersection of khalfspaces. We conjectured that all polytopes are epotylops, and all epotylops are polytopes.

Optimization / Linear Programming

In a polytope, how do we "optimize" a specific direction, i.e., given a direction, how do we maximize $x \cdot y$ for all $x \in P$? An intuitive approach is to turn the polytope so that the given direction faces down, then drop a ball, which will slide to the maximal point under the influence of gravity. We are normally given the polytope as an epotylop, so we work with hyperplanes/facets. In order to apply this in practice, we started on a point in the polytope, and always moved along the edge (1-face) that was closest to the given direction, such that the next point was always at the intersection of that edge with the first facet. Once there are no such edges (all adjacent edges point away from the given direction), the point is maximal. One problem is choosing an initial point, which can be solved by applying the algorithm to additional augmentation variables, which have an obvious initial point, and can give an an initial point for the original polytope once their cost function is minimized.

Ultimate Symmetry

We tried to define "the ultimate symmetry", such that polytopes P having the ultimate symmetry had their vertices on a d-sphere, had all k-faces congruent, had a bijective "rotation" sending any given face to any other given face of the same dimension, and had all normal lines through the "center" of each face meeting at the "center" of the polytope.

Glamour Deceit

The glamour deceit $gd(P) = conv(\{\vec{y} | H(\vec{y}, 1) \cap P \text{ is a facet }\})$. This is kind of an "affine dual".

Conjectures: $gd(P) = \bigcap_{y \in ext(P)} K(\vec{y}, 1)$, P, $gd(P)$ are combinatorial duals, and when it works, $gd(gd(P)) = P$. The glamour deceit currently only seems to work for d-polytopes in d-space containing the origin.

Summary of Tom and Hannah's Class - Dynamics Fun Time!

By: Seri Choi

Iterating Functions

Iterating functions can be written as,

$$x_0,\ f(x_0),\ f(f(x_0)),\ f(f(f(x_0))),\ f^{(4)}(x_0),\ \ldots$$

The following conditions can happen in iterating functions:

1) cycle: $f^{(n)}(x_0) = f^{(m)}(x_0)$, where $n > m$.
    a) fixed point: $f(x_0) = x_0$ or $f^{(n)}(x_0) = f^{(n+1)}(x_0)$.
        i) attractive fixed point
        ii) repelling fixed point
2) explode: "goes to ∞", meaning no upper bound
3) sequence of iterates converges to a point or to a cycle.

Metric Spaces: X set of points

Metric spaces are defined by a distance function

These are desired properties:

1) symmetry: $d(a,b) = d(b,a)$
2) $d: X * X \rightarrow \Re \geq 0$
3) $d(a,a) = 0$
4) triangle inequality: $d(a,b) + d(b,c) \geq d(a,c)$
5) if $d(a,b) = 0$, then $a = b$

Open Ball

The open ball around a point p with radius r is denoted $B(p,r)$ and is defined by

$B(p,r) = \{x \in X : d(p,x) < r\}$

Say we have a sequence $a_1,\ a_2,\ a_3,\ \ldots$ in X. When does $\{a_i\}$ converge to $p \in X$?

1) For every r, incluing r very small, there is an M, such that for all $i \geq M$, $a_i \in B(p,r)$.
2) Equivalently, every open ball around p excludes only finitely many points of the sequence.

Cantor Set

The Cantor set is the set of all the points in $[0,1]$ that are not removed at the nth step for any n. We can construct Cantor set by starting with the closed interval $[0,1]$ and removing the "middle third", which is the open interval $(\frac{1}{3}, \frac{2}{3})$. Repeat removing the open middle third of each remaining interval.

The Cantor set satisfies the following equation:
$f(S) = \frac{1}{3}(S) \cup (\frac{1}{3}S + \frac{2}{3})$

<u>Shift map:</u> x $\in C$
S$((.a_1\ a_2\ a_3 \ldots)_3) = (.a_2\ a_3\ a_4 \ldots)_3)$
There are countably-infinite many periodic points in C under the shift map
If x = $(.0\ 2\ 00\ 02\ 20\ 22\ 000\ 002\ 020 \ldots)_3$, then this x has an actual <u>aperiodic</u> orbit under the shift map S.

Affine Transformation
$f: V \rightarrow W$ is affine if $f(\overline{v})$ is the sum of a given linear transformation and a constant.
$f(\overline{v}) = t(\overline{v}) + \overline{p}$, where $t: V \rightarrow W$ is linear and $\overline{p}$ is constant.

Self Similar
A set S in a metric space (x, d) is self similar if S is the union of a finite number of subsets, S_1 through S_k, where $S_i = f_i(S)$ and f_i is an affine transformation. Sierpinski triangle is an example.

Similarity Dimension
Object has N self-similar pieces each scaled by a ratio of r. Then, the similarity dimension D is the number that satisfies $N = (\frac{1}{r})^D$.

Converging Limit
In a metric space X, the sequence x_i converges to a limit x if for all $\xi > 0$, there is M, such that for all $\iota > M$, $d(x_i, x) < \xi$.

Cauchy Sequence
A Cauchy sequence is a sequence y_i in X, such that for all $\xi > 0$, there is a M, such that for all $i, j > M$, $d(x_i, x_j) < \xi$.

Complete
The metric space X is called complete if every Cauchy sequence in X has a limit.
e.g. $\Re$ is a complete metric space.

Closed / Open
A subset S of a metric space X is closed, if for every convergent sequence $\{x_i\} \subseteq S$, limit of x_i is in S.
A subset S of a metric space X is open, if for all $x \in S$, there exists $\xi > 0$, such that $B(x, \xi) \subseteq S$.

*A set is closed if and only if its complement in X is open.

Dense

A set S is dense in an another set C if for all $x \in C$ and for all $\xi > 0$, $B(x, \xi)$ contains at least one point of S.

Continuity

- X, Y metric spaces, $f : X \to Y$ is continuous.
  - if $\{x_i\}$ is a convergent sequence in X, $x_i \to x$, then $\lim f(x_i) = f(x)$.
- For all $x \in X$ and all $\xi > 0$, there exists $\delta > 0$ such that $\{f(a) \mid a \in B(x, \delta)\} \subseteq B(f(x), \xi)$
- $f : X \to Y$, if $x_i \to x$ in X, then $f(x_i) \to f(x)$ in Y

Contraction Mappings

X, Y metric spaces $f : X \to Y$ a function $m \in \Re > 0$

We say f is <u>m-Lipschitz</u> if for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq m * d_X(x_1, x_2)$

Assume X is complete. If $f : X \to X$ is m-Lipschitz and $m < 1$, then f has an attractive fixed point

- For any $x \in X$, the orbit $x, f(x), \ldots, f^{(n)}(x), \ldots$ is a Cauchy sequence.
- If $f^{(n)}(x) \to y$, then y is a fixed point
- If $f^{(n)}(x_1) \to y$, then $f^{(n)}(x_2) \to y$, too.

Turnip Code Stuff

Chaos Algorithm for IFS(Iterated Functions Systems)s (Michael Barnsley)

Collection of affine maps F_i, $F_i : \Re^2 \to \Re^2$ all contraction mapping (i.e. m-Lipschitz)

IFS is $F = \bigcup_{i=1}^{n} F_i$

The definition of Chaos Algorithm is, take one initial point x_0. Let $x_k = F_i(x_{k-1})$, where F_i is chosen at "random"

The following conditions are needed on F_i to get the Chaos Algorithm to "work"

- each F_i must dilate by a factor less than 1 (dilate by S, want $0 \leq S < 1$)
- want the "random orbit" to be dense in our fractal

Dilations on an arbitrary metric space (X, d)

$f : X \to X$, such that $d(f(p), f(q)) = S * d(p, q)$, for $0 \leq S < 1$.