Monday Daily Gather: When is a Circle not a Circle?

By: Seri

On Monday, Amy Ksir from US Naval Academy came as a guest speaker to give us a talk about the graphs of equations in different number systems.

We started the daily gather by listing the things we know about the properties of circles. We agreed on the following facts:

- Constant curvature

- Every point has a fixed distance from the center

- no 3 points on the circle are collinear

- Has a radius & a center

- 0 eccentricity

- In 2D, 3 non-collinear points determine a circle

- Symmetric about center

- maximum area for a given perimeter

- disk vs. circle
  - disk: filled-in circle; 2D shape in 2D space
  - circle: blank circle; 1D shape in 2D space

Then, we combined these properties to find the definition of a circle. We had a debate whether a circle consists of a set of coplanar points or not, and we eventually came up with the following definition.

Definition of a Circle (Geometry Part)
A circle is a 1-dimensional shape consisting of the set of all points in a given plane that are a fixed distance from the center.

We could approach the definition in a different way, too.
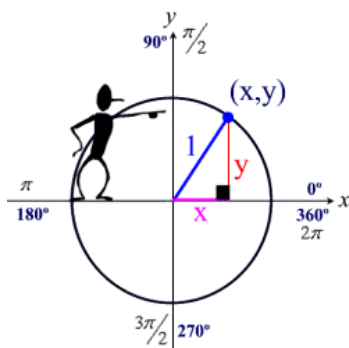
A circle can be described with this equation:
$$(x - s)^2 + (y - k)^2 = r^2,$$
where r is the radius and (s, k) is the center.
For example, let the circle on the left be centered at the origin and have a radius of 1. This circle can be described with,
$$\{(x, y) : x^2 + y^2 = 1\}.$$
An interesting idea to consider is when the center is not on the same plane as the circle.

http://www.regentsprep.org/Regents/math/algtrig/ATT5/unitcircle.htm

<u>Definition of a Circle (Algebra Part)</u>

Given a Cartesian plane, a point (s,t) on the plane, and a positive real number r,

$\{(x,y) : (x-s)^2 + (y-k)^2 = r^2\}$ is the circle on that plane with that center and radius.

A question was posed: How many intersections do you get when you intersect a line and a circle? We quickly figured out that there should be 0, 1, or 2 intersection points. We came up with some examples to see if the answer was true.

Equation for a line can be written as, $ax + by + c = 0$.

For the first example, we looked at 2 intersection points between a circle and a line:

  Suppose there exists a circle, $x^2 + y^2 = 1$, and a line, $x = y$.

  The intersection would be $\{(x,y) : x^2 + y^2 = 1 \ and \ x = y\}$.

  If (x,y) is in the intersection, then $2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \frac{1}{\sqrt{2}} \ or \ -\frac{1}{\sqrt{2}}$.

  The intersection points must be $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \ or \ (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

An example for 1 intersection point was:

  Suppose there exists a circle, $x^2 + y^2 = 1$, and a line, $y = 1$.

  $x^2 + 1 = 1 \Rightarrow x^2 = 0 \Rightarrow x = 0$.

  Therefore, the intersection point must be $(0, 1)$.

Another example we came up with was:

  Suppose there exists a circle, $x^2 + y^2 = 1$, and a line, $y = 2$.

  $x^2 + 4 = 1 \Rightarrow x^2 = -3 \Rightarrow x = \pm\sqrt{3}i$.

  Therefore, the intersection points must be $(\sqrt{3}i, \ 2) \ or \ (-\sqrt{3}i, \ 2)$.

Is the last example an example of 0 intersection or 2 intersection points? (2 intersection points)

Then, this question was asked:

Consider the set $\{(x,y) : x^2 + y^2 = 64\}$. If we allowed x and y to be complex numbers, what the heck does this look like?

The answers we came up with were this.

1. Choose a point for x^2 reflect through 32, and solve for roots.
2. Fix y, solve for x. (or vice versa)
3. Graph with 2 real axes, and 1 imaginary axis.

*Also, we need to think about both positive and negative square roots.
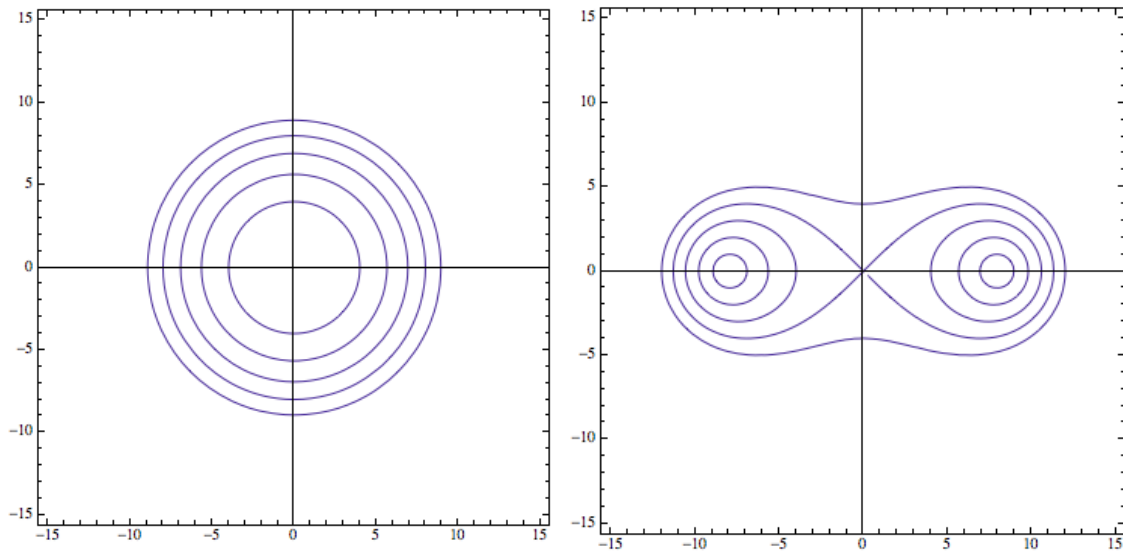
We were given another question: For a given value of y, how many x-values are there? We found out that there would be 2 x-values, but why?

These are some cool analysis to do:

1. Infinity points form a sphere?!

Consider the set $\{(x,y) : x^2 + y^2 = 64\}$. If $y = ix$, then there would be no points because $0 \neq 64$.

$$1 + \left(\tfrac{y}{x}\right)^2 = \tfrac{64}{x^2}$$
$$0 = \tfrac{64}{x^2}$$

So, there are infinite number of points that satisfy x.

2. $y^2 = x(x+1)(x-1)$

Tuesday Daily Gather: How to Win a Game of Chess Against Two Game Grandmasters (a.k.a Conway's Hackenstring)

By: Hannah and Josh

Daniel began this week's Tuesday Daily Gather by considering which games we can analyze and strategize with mathematical rigor. The conditions: two players, deterministic (that is, luck is not a factor), perfect info (each player is aware of all previous moves), and partial (each player has access to a specific set of moves, perhaps in the form of "game pieces"). Tic-tac-toe, Checkers, and Chess fit these conditions whereas 2048 (fails two player condition), Set (fails deterministic condition), Mafia (fails perfect info condition), and Nim (fails partial condition) do not.

We now turn to the game that captured our attention: Conway's Hackenstring. Hackenstring is a game for two players. The game board consists of stacks of blocks labeled, in our case, K for Kevin and S for Sugoi. On his turn, a player removes any of his own blocks from any of the stacks; accordingly, all blocks above the removed tile "topple" over (i.e. are removed from the gameboard). The player who cannot make a move on their turn loses. Here is an example of a Hackenstring game—say it is Kevin's turn. He has two options (the K in the first stack or in the second):
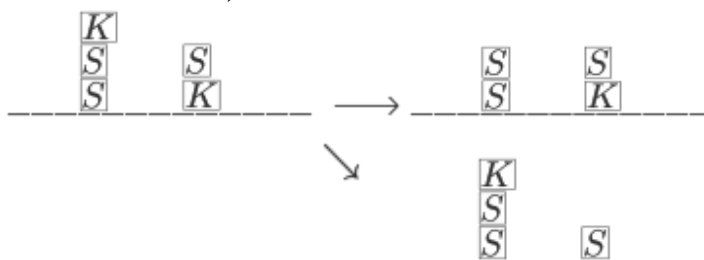
It is apparent that no matter how Kevin plays, Sugoi will always remove the last block and henceforth win. Thus, we call this a K_neg game since Kevin loses regardless of which player goes first. (Similarly, a K_pos game is a game in which Kevin wins regardless of which player goes first.)

Who wins this game?
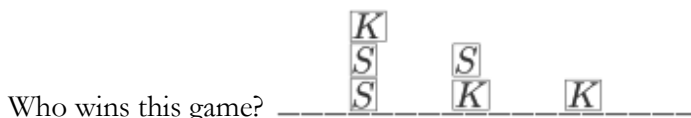
If Kevin plays first then Sugoi wins. If Sugoi plays first then Kevin wins. It seems this game is neither K_pos or K_neg! Thus, we call this the Zero game. A natural question to ask is

whether or not this "Zero game" satisfies the usual properties of a zero element. Answer: yes! First, we define the addition of games as placing the games side by side on a single board. (We see that both associativity and commutativity hold).

Does a game G have an inverse game (that is, a game "-G" such that when G and -G are placed on the same board, we get the Zero game)? Also yes! We form the game -G by swapping all $\boxed{K}$ blocks and $\boxed{S}$ blocks in G. Since the second player can always use the mirror strategy and force the first player to lose, we have that -G + G = 0, and an inverse game exists.

We can also define the "magnitude" of a game as the amount of blocks by which a player wins. For instance,

$$\underline{\quad\quad\boxed{K}\quad\quad} = \text{+1 since Kevin always wins by one block}$$

$$\underline{\quad\quad\genfrac{}{}{0pt}{}{\boxed{K}}{\boxed{K}}\quad\quad} = \text{+2}$$

and in general,

$$\underline{\quad\quad\genfrac{}{}{0pt}{}{\boxed{K}}{\genfrac{}{}{0pt}{}{\vdots}{\genfrac{}{}{0pt}{}{\boxed{K}}{\boxed{K}}}}\quad\quad} = \text{+n, where there are n } \boxed{K} \text{ blocks.}$$

Amazingly, we can also define fractional games. For instance $\underline{\quad\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\boxed{K}}\quad\quad}$ is the +½

game. We can check this by confirming that $\underline{\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\boxed{K}}\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\boxed{K}}\quad\boxed{S}\quad}$ is the Zero game.

(G+G+(-1)=0 ⇒ G=+½). Generalizing, we have $\underline{\quad\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\boxed{S}}{\boxed{K}}}\quad\quad}$ is the +½^n game, where

there are n $\boxed{S}$ blocks above a single $\boxed{K}$ block. We can prove this in the same manner as in the n=2 case, by checking the following as the zero game:

$$\underline{\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\vdots}{\boxed{K}}}}\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\vdots}{\boxed{K}}}}\quad\genfrac{}{}{0pt}{}{\boxed{S}}{\genfrac{}{}{0pt}{}{\vdots}{\boxed{K}}}\quad}$$

In Daniel's Daily Gather, we began to uncover the surprising relationship between a seemingly simple combinatorial game and the numbers we know and love.

Wednesday Daily Gather: Vanishing Sets of Polynomials

By: Nathan

Nate talks about polynomials and their roots.
If P is a polynomial, Z(p) is the set of points where P takes the value 0. For instance,
$P(x)=x^2-1$
$Z(p)=\{-1,1\}$

In general,
$P(x)=a_0+a_1x+a_2x^2+...+a_kx^k$

Dennis claims that any finite set of real numbers can be the vanishing set of a polynomial.

Heesoo claims that any polynomial of degree k has at most k roots:
$(x-x_1)(x-x_2)(x-x_3)...(x-x_k)=0$
$x_1,x_2...x_k$ are the roots of this equation.

Theorem: A polynomial of degree k has at most k roots
Proof by Induction:
Base case: k=1
$ax+b=0$
$x=-b/a$
Inductive Step:
Let $P(x)$ be a polynomial of degree k+1.
If $P(x)$ has no roots, then 0<k+1 and we are done.
If $P(x)$ has a root, say x_0, then write $P(x)$ as $(x-x_0)Q(x)$ where $Q(x)$ has degree k. We assume all polynomials of degree k has at most k roots, so it has at most k roots, thus P has at most k+1.
Now consider polynomials of two variables.
$P(x,y)=a_{0,0}+a_{1,0}x+a_{0,1}y+a_{1,1}xy...$
Define the degree of P to be the maximum m+n where the terms are written as x^my^n.

Q: Is there a nonzero polynomial vanishing at $\{(1,2),(2,4),(3,8)...(2014,2^{2014})\}$? What is the minimum degree of this polynomial?
Jån gives 1007 by lines
GPH gives 806 with conics (five points form a conic)

Thursday Daily Gather: "Vanishing" Sets of "Polynomials"

By: Vandana, with graphs by Dennis

On Wednesday, for the Daily Gather, Cynthia Vinzant gave a talk in which we defined two functions and discovered that many of the properties which hold for functions we know hold for these as well.

For $x, y \in R \cup \{-\infty\}$, define:

$$x \oplus y = \max\{x, y\}$$

$$x \odot y = x + y$$

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c) \text{ because a + max(b,c) = max(a+b, a+c)}$$
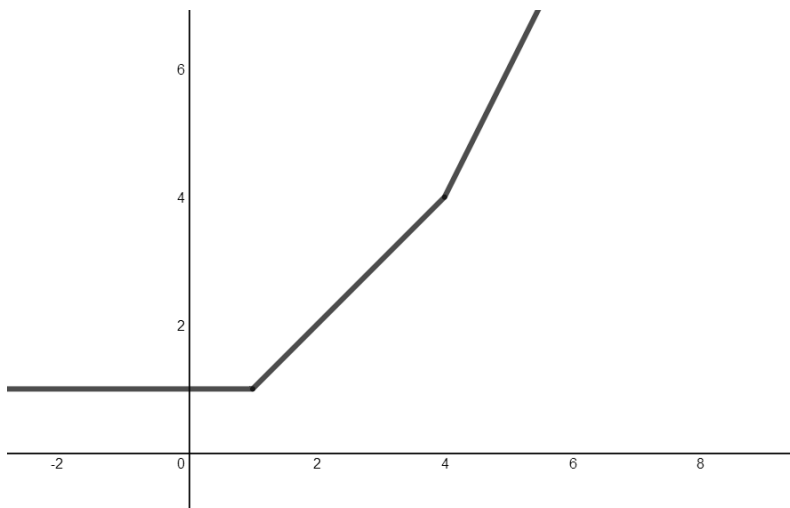
$$x \odot 0 = x$$

$$x \oplus -\infty = x$$

WARNING: $1 \oplus 3 = 3$ *and* $2 \oplus 3 = 3$, *no* $\ominus$

"Polynomials"

$$P(x) = a_0 \oplus (a_1 \odot x) \oplus (a_2 \odot x \odot x) \oplus ... \oplus (a_k \odot x \odot ... \odot x)$$
$$= a_0 \oplus (a_1 \odot x) \oplus (a_2 \odot 2x) \oplus ... \oplus (a_k \odot kx)$$

ex. $1 \oplus x \oplus ((-4) \odot 2x)$

"Z(p)" = {r at which the slope of p(x) changes}

Q: How many roots does $(a \odot x) \oplus b$ have?

    A: one; x = b-a

Q: How many roots does a quadratic have?   (ex. $(a \odot 2x) \oplus (b \odot x) \oplus c$

    A: at most two

Question to think about: Is there a way to define multiplicity so that a quadratic has two roots? (change in slope?)

WHY ARE WE DOING THIS?

$$t^a \cdot t^b = t^{a+b}$$

$$t^a \cdot t^b \approx t^{max(a,b)} \text{ for t infinitely large}$$

$$log_t(t^a \cdot t^b) = a+b$$

$$\lim_{t \to \infty} log_t(t^a + t^b) = max(a,b)$$

Take $1 \oplus x \oplus (2x \odot (-4))$

$$5t + ex + \sqrt{2}t^{-4}x^2 = P(x)$$

$$P(1) = 5t - e + \sqrt{2}t^{-4} > 0, \; P(t^2) < 0, \; P(t^6) > 0$$

Up a Notch (precisely one - no more, no less :)) - P in two variables

$$P(x,y) = \sum_{j,k}(a_{jk} \odot jx \odot ky)$$

"Z(p)" = {(x,y) where this max is achieved $\geq$ twice.

Another question to think about: Do every two lines intersect in a point (except for parallel lines)?

Let's draw a conic.

$$2x \oplus (x \odot y \odot 1) \oplus 2y \oplus (x \odot 1) \oplus (y \odot 1) \oplus 0$$