WEEK 2

2014

RECORD OF MATHEMATICS

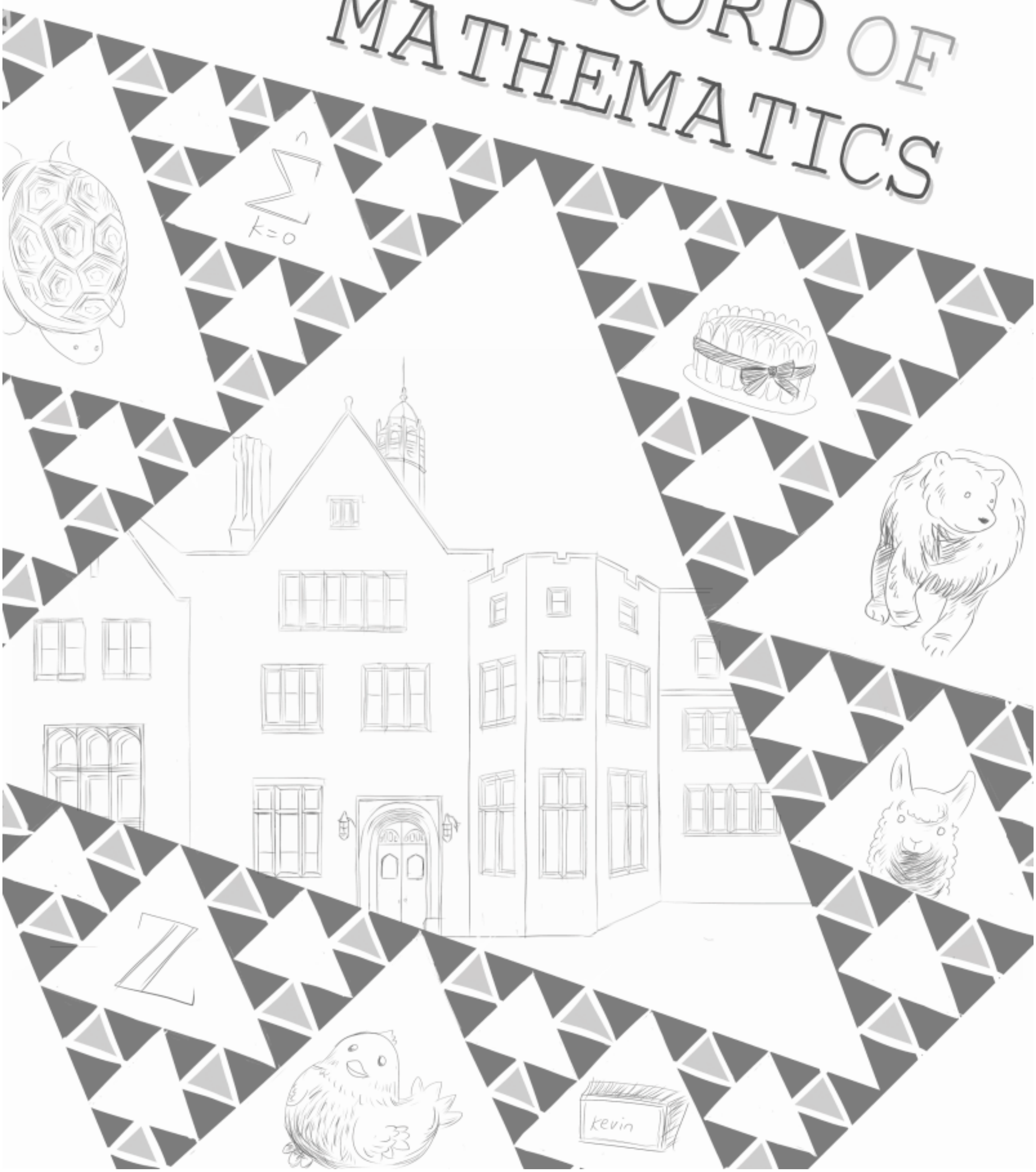

Table of Contents

Acknowledgements

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Assistant Instructors: Hannah, Miles, and Nate
PRiME: Corrine

"We are not the sort of program where people eat slugs."
-Corrine

Calendar for Week of Chaos at MathILy:

A reminder of our weekday-ly schedule:
Breakfast 8:15–8:55 at Erdman
Morning Classes in Park 336, 337, 328
 Chaos 1, 9:00–10:15
 Chaos 2, 10:25–11:40
 Chaos 3, 11:50–1:00
Lunch 1:05–1:30 at Erdman
Daily Gather 4:30–5:30 in Park 336
Dinner 5:35–6:30 at Erdman
Evening Classes in Park 336, 337, 328
 Chaos 4, 7:00–8:25
 Chaos 5, 8:35–10:00

Special things:
- Note the separate Week of Chaos Class Grid to figure out where you should go when.
- On Thursday, read Branch Class descriptions and submit preference forms by the start of Friday evening classes.

Daily Gathers:
Monday: Amy Ksir (US Naval Academy)—When is a circle not a circle?
 The geometry of equations in different number systems.
Tuesday: Daniel Studenmund (Univ. of Utah)—Can you add SET to Settlers of Catan?
 No. Or, at least, probably not. But we will add other games! It will be fun.
Wednesday: Nate—Vanishing sets of polynomials
 Now you see them... now you don't.
Thursday: Cynthia Vinzant (MSRI and NCSU)—Exponentially improving arithmetic
 Are you tired of the usual rules of arithmetic? Multiplying two numbers takes so much time! Let's say instead that sum of two numbers is their maximum and the product of two numbers is their sum. Many of your favorite theorems from geometry are still true, plus a few more.
Friday: More Math Movies! and pre-Philadelphia trip discussion

Saturday schedule:
Breakfast 8:15–8:55 at Erdman
Let's go to Philadelphia!
 Meet at Erdman ready to leave for the entire day at 9AM promptly. No, really. Promptly.
 Then meet at Market East around 9PM to go home.

Sunday (National Ice Cream Day) schedule:
Meet at 2pm outside of Denbigh to go for ice cream!

Emergency Triangles

Overwhelmed by so much total angle? We can reduce the burden—all the way to 0° if you want. Rush delivery of hyperbolic triangles available: call 725-6145.

Root Class Summary: sarah-marie's Class

by Nathan

Note-Taking Algorithm

Once again, the class began the week by assigning note-takers. However, the algorithm used in this week's diagram was slightly different than that of week 1. It was allowed for a line to go off of the diagram and return on the opposite side. Also, a new "R" category was introduced, which denoted the person who was to write the ROM article summarizing the week's root classes.

RARs

The MathILy students were introduced to RARs, known formally as groups. They have the properties of closure, associativity, inverse, and identity. The students thought of several examples of these objects such as: the integers under addition, the rationals without zero under multiplication, and more.

Vrooms

Vrooms, formally known as vector spaces, have quite a bit in common with RARs. They contain both vectors and scalars and follow several axioms. They contain both a set of vectors and of scalars. Also, they satisfy several properties under vector addition and scalar multiplication.

General Position

AI Hannah defined a set of at least d hyperplanes to be in general position if every d hyperplanes intersect at exactly 1 point (unicorn) and no d+1 hyperplanes intersect at a point (dragon). This concept was particularly useful for further analyzing the tiramisu problem.

Nowé Definitions

Two nodes are called adjacent if there is an edge connecting them. A node and an edge are called incident if the edge if the edge is attached to the node. A subnowè is a nowè is a set of edges and nodes part of the larger nowè.

Adjacency Matrices

An adjacency matrix is a matrix that tells you whether two nodes are adjacent. The MathILy students were challenged to prove whether powers of the adjacency matrix can be used to count strolls between two nodes.

Probability

A problem involving Nate's socks was presented to illustrate various ideas in probability theory. A probability space is a set in which each element is called a state and each group of elements is called an event. A mapping from the power set of a probability space to the real numbers from 0 to 1 where the probability of the union of all states is 1 and the probability of the null set is 0 is called a probability measure. These things were then related to vrooms.

Matrices as Functions

The students then analyzed functions where the input and output were both vectors and the function was defined by a matrix. The students looked at things such as bijectivity, the identity function, inverse matrices, and more.

Hilbert Hotel

Then, sarah-marie began reading from a mathematical book which detailed the holiday adventures of the Smith family. They decided to vacation to an infinite hotel and the book illustrated various counter-intuitive things the can happen when dealing with infinity.

The Absentee Club

At the Hilbert Hotel, one of the infinitely many clubs was the absentee club. This club consisted of all people who do not belong to the club that meets in their room. However, there is a contradiction in regard to the people who live in the room that the absentee club meets in. Thus, the absentee club cannot meet in a room. This same line of reasoning can be applied to prove that the power set of the natural numbers is uncountable.

Hyperbear Neutralization

Miles and the starfleet have once again developed a new way to defend humanity from the deadly threat of n-dimensional bears. The newest weapon is a panel that works to fight against the bears. However, due to budget constraints, no wires are to cross. This problem was quickly solved for the specific example Miles presents, however it was generalized with the inequality $e \leq 3(n-2)$, which can show that certain graphs are impossible to solve, but a positive result does not guarantee a solution.

Summary of Tom and Nate's Class

By: Heesoo Kim

Vector Space

A vector space must satisfy the following conditions: (1) $\vec{} \in S$, (2) $if\ \vec{v} \in S, then\ a\vec{v} \in S\ \forall a \in \mathbb{R}$, and (3) $if\ \vec{u}, \vec{v} \in S, then\ \vec{u} + \vec{v} \in S$.

Dimension

If two vector spaces are linearly isomorphic, then they have the same dimension. If V is isomorphic to $\mathbb{R}^n$, it has n dimensions. Also, for a given vector space V, if B_1 and B_2 are two bases of V, then $|B_1|=|B_2|$.

Gruppa

A gruppa set, Γ, under an operation * must satisfy the following conditions: (1) (a * b) * c = a * (b * c), (2) $\exists e \in \Gamma$ such that $a * e = e * a = a, \forall a \in \Gamma$, and (3) $\forall a\ \exists b$ such that a * b = b* a = e.

Expected Value

If Nate chooses 2k socks to bring out of n pairs of argyle socks, how many pairs did he end up with? A random variable is any function $X: S \to \mathbb{R}$, and an expected value for a random variable is the weighted average of all values that variable may have: $E(x) = \sum_{i=1}^{|S|} \left(X(S_i) * \mathbb{P}(S_i)\right),\ S = set\ of\ states.$

Capitalistic Pig Gruppa

Let Γ be a gruppa such that $\forall\ a \in \Gamma, \mathrm{a} * \mathrm{a} = \mathrm{a}^2 = \mathrm{e}$. We proved that Γ is a capitalistic pig (aka commutative).

Binomial Theorem

Binomial Theorem states that $(a+b)^n = \binom{n}{} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + \binom{n}{n} a^0 b^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-1} b^i$.

PIE

Principle of Inclusion-Exclusion (PIE, or aka Cake) allows us to calculate the total number of elements in a union of non-disjoint sets by alternating over-counting and under-counting.

Independent Events

Independent events are two events in some probabilistic space that do not influence each other. Meaning, if we have two events A and B, A happening does not affect the probability of B happening.

Reflection

If it flips about something, it is a reflection! For example, multiplying by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ flips the function about x-axis.

Matrices

Matrices are multiplicatively associative because composition of functions is associative. We can also stretch a function by multiplying it with certain matrices.

Eigenvalue & Eigenvector

Eigenvector of a square matrix A multiplied by some non-zero vector always give the same value called eigenvalue. We denote the eigenvalue with λ. Basically, we get this: $Av = \lambda v$. Do all matrices have eigenvalues? Yes! Granted, some have complex numbers as their eigenvalues, but that's OK, because all we need to add is complex scalar multiplication to the vectors!

Dihedral Gruppa, D_n

These are symmetries of the regular n-gon. We conjectured, and later proved, that the size of the subgruppa must divide the size of the gruppa.

Llama Hiring

We proved Patrick's conjecture that if every subset S of llamas satisfies $|S| \leq |N(S)|$, then a llama-perfect matching exists, meaning every llama gets the job (s)he wanted. In our inductive proof, we considered the equality case and strict inequality case separately and finally proved Patrick's awesome conjecture.

Planar Graph

Graphs that can be drawn on a plane with no edges crossing is called planar graphs. There are cool theorems associated with this, 4-color theorem among them.

Four Color Theorem

This theorem states that for any planar graph, at most four colors are needed to color the vertices such that no two vertices that are connected by an edge may share a color.

Euler's Formula

Euler's formula states that V - E + F = 2 where V = # of vertices, E = # of edges, and F = # of faces. We proved this formula in both combinatorical and geometrical way.

Maximal Planar Graph

This is just a planar graph that has maximum # of edges for certain number of vertices without any loops or multiple edges connecting two vertices.

Deficiency

Consider a vertex of a 3D shape such that the vertex is formed through intersection of multiple planes. The sum of angle of individual 2D shapes that surround the vertex subtracted from 2π is the deficiency of the corner.

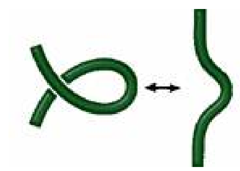
Monday Daily Gather:
Vaughan's Surf-Tastic Polynomial

by Cat

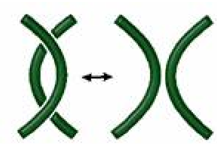
On Monday, Emily Peters from Loyola University Chicago talked to us about knots, or closed curves in 3D space. Emily taught us some basic ways to to distinguish between different knots. We first decided that two diagrams represent the same knot if the knot in one diagram can be rearranged to get the knot in the other by a sequence of Reidemeister moves:

Images from: http://www.oglethorpe.edu/faculty/~j_nardo/knots/

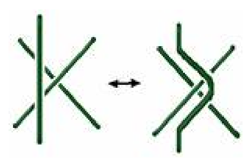
1. Take out (or put in) a simple twist in the knot:



2. Add or remove two crossings (lay one strand over another):



3. Slide a strand from one side of a crossing to the other:



We quickly found that it can be pretty challenging to tell two knots apart, especially when the knots are really complicated. So we tried to come up with something, called an invariant, that could tell us whether two knots were the same. An invariant of a knot diagram is a map such that Reidemeister 1, 2, and 3 don't change the value of the invariant.

Emily eventually suggested that we use polynomials as our invariant. She introduced Kauffman Brackets, denoted $< k >$, where k is the knot. but we quickly ran into some difficulties calculating the values of the polynomial with Reidemeister 1.

Emily then suggested that we define a different invariant. We added orientation to the strands in the knot, and assigned a value to each of the two different kinds of orientation on a crossing. T *twist* of an oriented knot was the sum of all the signs of all its crossings.

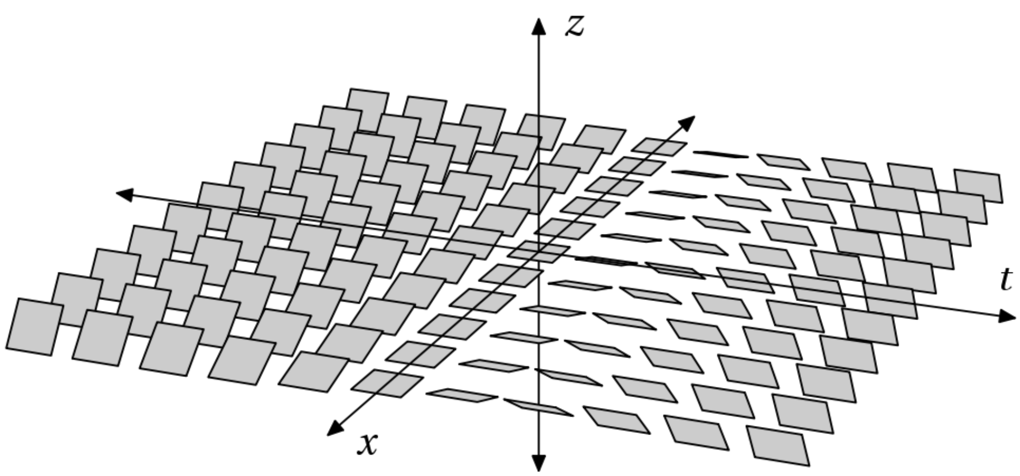
Emily then introduced the Jones polynomial, which takes into account the orientation of the knot. The Jones polynomial is $V(k) = -A^{-3(twist(k))} < k >$. Using the Jones polynomial, the issue with Reidemeister 1 was no longer an issue. With Jones polynomial, we were finally able to tell if two knots were the same.

Tuesday Daily Gather:

by Gideon

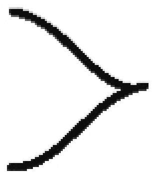
Professor Joshua Sabloff came in to tell us about Legendrian knots. He posed the following scenario: you're riding a unicycle down the street, and you find a nice spot to park. How do you continue?

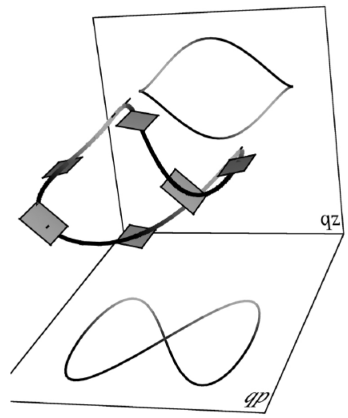
We constructed a mathematical model of the unicycle with respect to the curb by giving the unicycle coordinates (x, t, z), with x representing the position along the curb, z the distance from the curb, and t the slope of the unicycle's wheel relative to the curb (projected onto the street). An added condition was that you cannot park perpendicular to the curb, in order to avoid infinite slope. We ignore other parameters, which is okay because mathematical models are necessarily lies. The only two allowed movements of the unicycle are rotations, which change t, and forward movements in the direction of the wheel. Any composition (linear combination) of these two is allowed as well, so the set of all possible motions at a point (x, t, z) is span{[0 1 0], [1 0 t]} (*******these vectors should be vertical), which forms a plane in R3. Because this is defined differently at any point, the set of all possible movements forms a plane field in R3.



(http://upload.wikimedia.org/wikipedia/en/thumb/f/f5/Standard_contact_structure.svg/1000px-Standard_contact_structure.svg.png

The question was posed: Given any two unicycle configurations, can you move from one to the other using allowable motions? Natasha proposed that, in the general case, the unicycle could just move toward the intersection of the lines created by the wheels, rotate once there, then move toward the other position. However, this is not a good model of real unicycle movement, and the movements aren't smooth. In order to have a smooth path, and to avoid vertical slope, we can reverse direction at a point. This path doesn't appear smooth in 2D, but this is only a projection of a smooth 3D curve onto the xz-plane.

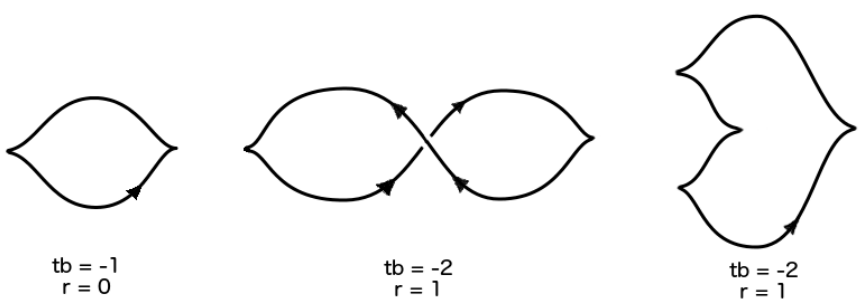




(http://cnx.org/content/m36041/latest/FrontRmoves1.png), (http://reu.dimacs.rutgers.edu/~maboemo/imgF.gif)

This defines a curve always laying tangent to a plane field, or a Legendrian curve. A Legendrian knot is a Legendrian curve that starts and ends in the same place. We can represent Legendrian curves by their projections on the xz-plane, where crossings can be distinguished because sections with negative slope are in front of the sections with positive slope when viewed in 3-dimensions. We say that two Legendrian knots are equivalent if one can be deformed into the other such that each intermediate knot is also Legendrian. Like other knots, equivalent Legendrian knots can be related by a series of Reidemeister moves (which are different from the standard Reidemeister moves). However, this is difficult, so we use invariants to tell them apart, some of which require an orientation for the knots. Some invariants:

1. "Ordinary" knot type
2. The Thurston-Bennequin number (tb), defined as the number of positive crossings minus the number of negative crossings minus the number of cusps.
3. The rotation number (r), defined as the number of down cusps minus the number of up cusps.



(Graphics created by Gideon)

All of these Legendrian knots are "ordinarily" unknots. Because the invariants have different values for the first knot, it is not equivalent to the other two. However, we cannot determine whether the other two are equivalent because their invariants are equal.

Thursday Daily Gather: Divisibility Rules by Vandana

Presentation by Rachel Kirsch

DR3

Claim: Let $n \in N$. Let $d_i, d_{i-1}, \ldots, d_i, d_0$ be the digits of n (Left-to-Right)

$3|n \Leftrightarrow 3|d_i + d_{i-1} + \ldots + d_1 + d_0$

Proof: $n = 10^i d_i + 10^{i-1} d_{i-1} + \ldots + 10\, d_1 + d_0$

$\equiv 1^i d_i + 1^{i-1} d_{i-1} + \ldots + 1\, d_1 + d_0 \;\; mod\, 3$ because $10 \equiv 1\; mod\, 3$

$\equiv d_i + d_{i-1} + \ldots + d_1 + d_0 \;\; mod\, 3$

$3|n \Leftrightarrow n \equiv 0\; mod\, 3 \Leftrightarrow d_i + d_{i-1} + \ldots + d_1 + d_0 \equiv 0\; mod\, 3 \Leftrightarrow 3|d_i + d_{i-1} + \ldots + d_1 + d_0$

DR9 Exactly the same process as for 3 because $10 \equiv 1\; mod\, 3$ and $10 \equiv 1\; mod\, 9$

DR37

Claim: Let $n \in N$. Split n into blocks of 3 consecutive digits, called $b_i, b_{i-1}, \ldots, b_1, b_0$

$37|n \Leftrightarrow 37|b_i + b_{i-1} + \ldots + b_1 + b_0$.

Proof: $37 \cdot 27 = 999 \Longrightarrow 1000 \equiv 1\; mod\, 37$

$n = 10^{3i} b_i + 10^{3(i-1)} b_{i-1} + \ldots + 10^3 b_1 + b_0$

$\equiv b_i + b_{i-1} + \ldots + b_1 + b_0 \;\; mod\, 37$

$37|n \Leftrightarrow n \equiv 0\; mod\, 37 \Leftrightarrow b_i + b_{i-1} + \ldots + b_1 + b_0 \equiv 0\; mod\, 37$

$\Leftrightarrow 37|b_i + b_{i-1} + \ldots + b_1 + b_0$

Are there any primes that function this way in blocks of 2? They must be 1 mod 100; p|99.

DRP

Claim: Let $n \in N$. If $10^k \equiv 1\; mod\, p$, split n into blocks of k consecutive digits and call them $b_i, b_{i-1}, \ldots, b_1, b_0$

$p|n \Leftrightarrow p|b_i + b_{i-1} + \ldots + b_1 + b_0$

Proof: $n = 10^{ki} b_i + 10^{k(i-1)} b_{i-1} + \ldots + 10^k b_1 + b_0$

$\equiv b_i + b_{i-1} + \ldots + b_1 + b_0 \;\; mod\, p$

Questions: What other primes are like 7 -- the block length is p-1: the highest possible?

For which primes p can we make a DRP? In other words, for which p is there a k such that $10^k \equiv 1\; mod\, p$?

Fermat's Little Theorem:

$a^p \equiv a\; mod\, p$ where $a \in Z$ and p is prime

$a = 10,\; for\; p \nmid a^{p-1} \equiv 1\; mod\, p$

For any prime $p \neq 2,5 \;\; 10^{p-1} \equiv 1\; mod\, p$. Every prime $p \neq 2,5\; has\; DR - p$.

$10 \equiv -1\; mod\, 11$ and $10^3 = -1\; mod\, 7,\; mod\, 11,\; and\; mod\, 13$

Friday Daily Gather: Models for Apportionment

by David

On Friday Jonah Ostroff graced us with a presentation about the long and tumultuous history of apportionment for the US House of Representatives. He presented a set of fictional states as an example and asked us to come up with various apportionment methods. It was soon that we found out that Heesoo and Åke are essentially the same as Alexander Hamilton and Thomas Jefferson respectively. In other words, Heesoo came up with Hamilton's method and Åke cam up with Jefferson's. Hamiltons method followed the following steps: compute an ideal number with the formula: $\frac{Population\ of\ the\ State * Total\ Seats}{Total\ Population}$, see what the seat shortage is, and round up the states whose ideal numbers have the highest fractional part. On the other hand, Jefferson's method followed the algorithm: choose a divisor, d, divide each population by d and round down, if that sums to the total required number of seats, that's great, if not pick another d and try again. Hamilton's method tends to favor smaller states while Jefferson's favors bigger states as demonstrated by the sample populations provided:

State	Population	Ideal number	Seats (Hamilton)	Seats (Jefferson d=.855M)
Alazona	1,450,000	1.45	2	1
Bassachussetts	3,400,000	3.40	3	3
Califlorida	5,150,000	5.15	4	6
Totals	10,000,000	10 (predetermined)	10	10

For purely political reasons, Jefferson's method was chosen, as the president at the time (George Washington) wanted to ensure that his home state of Virginia would be as overrepresented as possible. Jefferson's method was used for 50 years until people realized that it regularly over represented certain states by as many as 2 representatives. John Quincy Adams proposed a similar method, except it rounded up instead of rounding down, however, it suffered from the opposite problem (under-representing some states by 2 representatives). 60 years later, Daniel Webster (AKA Jo) came up with a method that rounds to the nearest integer, thus avoiding the favoring of larger or smaller states. To avoid issues, congress decided to pick a total number of seats where Hamilton's and Webster's lined up. However, it was promptly discovered that adding a seat to the total can occasionally decrease the amount of seats a state has (known as the Alabama Paradox). Ultimately, after much debate congress decided to use the Huntington-Hill Method which geometrically rounds (finds the lowest percent difference).